ON THE ORIENTATION OF DUST GRAINS IN INTERSTELLAR SPACE

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ABSTRACT

The observed polarization of starlight is regarded as being due to absorption by interstellar dust. A mechanism for alignment of the dust grains, due to Davis and Greenstein, is analyzed. A dissipative torque, due to paramagnetic relaxation in the grains, tends to line up a spinning dust grain in the interstellar magnetic field. This aligning torque is opposed by the randomizing effect of collisions with interstellar hydrogen. The aim of this paper is to find the distribution of orientations which results. The analysis is carried out through use of the Fokker-Planck equation. The solution is carried to completion for the case of small magnetic fields and nearly spherical dust grains, and the resulting polarization parameter is found. The spectrum of relaxation times due to the collisions is also found.

CONTENTS

I. Introduction

The polarization of starlight was first discovered in 1949 (1). It has been found that there is a correlation of the planes of polarization of the light from individual stars over considerable areas of the sky. There is also a correlation between the degree of polarization and the magnitude of the interstellar reddening. These observations are almost universally regarded as showing that the polarization is due to the interstellar dust. A non-spherical dust grain will in general scatter and absorb light of one plane of polarization more than another. If there is some mechanism to partially align these dust grains, then the transmitted light will be partially polarized, the amount being related to the total extinction. Since the short wave lengths are absorbed more than the long, this extinction can be observed as reddening. Various mechanisms have been proposed that might align the grains. This paper will analyze some aspects of one due to Davis and Greenstein.(2) The necessary anisotropy in space is here that of the interstellar magnetic field. With this model, then, some information can be obtained about the magnetic field.

Data on the reddening of starlight show that the

dust grains to be considered probably have radii in the range of (10^{-5})cm. Platt (3) has proposed a mechanism involving the quantum mechanical effects due to unfilled energy bands which requires much smaller grains. The model analyzed here, however, is based on classical absorption and scattering of light, and grains of about (10^{-5})cm. will be considered. In the details of the computation they will be taken to be spheroids. The alignment mechanism of Davis and Greenstein, one of paramagnetic relaxation, produces a dissipative torque which tends to align the angular momentum of the grain parallel to the applied magnetic field and the long axis of the grain perpendicular to this direction. Opposing this aligning mechanism will be the randomizing effect of impacts from the hydrogen atoms in the interstellar gas clouds. The aim of this analysis will be to find the equilibrium distribution of grain orientations produced by these two opposing effects. This is carried to completion for the case of small magnetic field and nearly spherical dust grains. The mathematical technique used is an approximate solution of the Fokker-Plank equation.

In the rest of this chapter some of the observational data is briefly given, the mechanical and optical model for the dust grains and the alignment mechanism

is described, and the Fokker-Planck equation is introduced. The dust grains are ultimately taken to be homogeneous ellipsoids of revolution, either prolate or oblate, and their absorption and scattering is taken to be given by the Rayleigh-Gans theory.(4) A collision with a hydrogen atom will be assumed to occur rapidly enough so that during the collision only the grains' angular momentum is changed, but not the grains' orientation in space. In Chapter II the differential equation for the entire alignment process is formulated. In Chapter III a perturbation solution of this equation for nearly spherical grains and small magnetic fields is found, and the results are used to deduce the polarization parameter for this case. Chapter IV contains a brief summary of the calculational procedures which may be useful as a guide to the rest of the paper. A summary of the results is given in Chapter V.

1. Polarization Observations

Let I_π and I_σ be the intensities of the light in the two perpendicular plans of polarization, with I_π in the plane of maximum intensity. In the model of Davis and Greenstein, I_π is the intensity of the light which has its electric vector in the plane formed by the galactic magnetic field direction and the direction of propagation. Then the degree of polarization is de-

fined by

$$\rho = \frac{I_\pi - I_\sigma}{I_\pi + I_\sigma} \qquad (1)$$

The observed values of ρ are of the order of .05. Astronomical results are usually expressed in magnitudes. The polarization in magnitudes is defined by

$$\Delta m_p = -(\tfrac{5}{2})\log_{10}(\frac{I_\sigma}{I_\pi}) \qquad (1a)$$

For small ρ, the two expressions for polarization are approximately related by

$$\Delta m_p = 2.2\ \rho \qquad (1b)$$

The polarization in magnitudes will be proportional to the amount of polarizing material along the light's path. This will also be true of the total extinction of the light. The total extinction, also in magnitudes, can be defined by

$$\Delta m = -(\tfrac{5}{2})\log_{10}\left[\frac{\text{Observed intensity}}{\text{Intensity if no absorbing dust were present}}\right] \qquad (2)$$

Because of the dependence of extinction on wave length, it is observed as a reddening of the light. This is usually described by a quantity called the color excess, E_1, which is about $\frac{1}{9}\ \Delta m$. Since it is independent of material along the path, the quantity $\Delta m_p/E_1$ or, for

small values of ρ, ρ/E_1 is of direct theoretical interest. The observed values of ρ/E_1 are of the order of .1.

The value .1 given for ρ/E_1 is an average value only and is given to illustrate the magnitude of the effect we are dealing with. Of much more astronomical importance is the maximum value, since we are then supposedly looking perpendicularly to the direction of the magnetic field at the location of the effective dust cloud. (see equation 3 below). Maximum values have been given as $\rho/E_1 \leq .25$. (5)

2. Mechanical and Optical Model

To calculate the expected polarization, or the ratio ρ/E_1, produced by the interstellar dust, it is necessary to

(a) assume some model for the grains and the torques which act on them,

(b) calculate the distribution of grain orientations resulting from the assumed model, and

(c) calculate the extinction vs plane of polarization for light incident on the oriented grains.

In this paper, (b) will be considered in detail and the results of Davis and Greenstein will be used for (a) (6) and (c) (7).

One of the principal results of the consideration of the grains' structure is that they will show the phenomena of paramagnetic relaxation.(8) The induced magnetization in a grain will have a finite relaxation time, so that in a time varying external field the magnetization will be out of phase with the field. In particular, if the grain is rotating with respect to the with respect to the field, the field and magnetization will not be parallel, thus producing a torque on the grain. The result is that this torque tends to make the angular momentum vector of the grain parallel to the magnetic field direction and the long axis of the grain perpendicular to this direction. This orienting torque will be opposed by the randomizing torques due to impacts of hydrogen atoms from the surrounding gas clouds on the grain.

The magnitude of these effects is small, so that the grain can be considered to be a free body with the torques producing slow changes in the free body rotation parameters. In figure 1, let $\underset{\sim}{B}$ be the magnetic field, $\underset{\sim}{H}$ the angular momentum of the grain, and $\hat{\underset{\sim}{A}}$ a unit vector along its symmetry axis. In the free rotation of a rigid body the angular momentum stays constant. Thus β, the angle between $\underset{\sim}{H}$ and $\underset{\sim}{B}$ stays constant. For a solid of rotational symmetry, the symmetry axis ro-

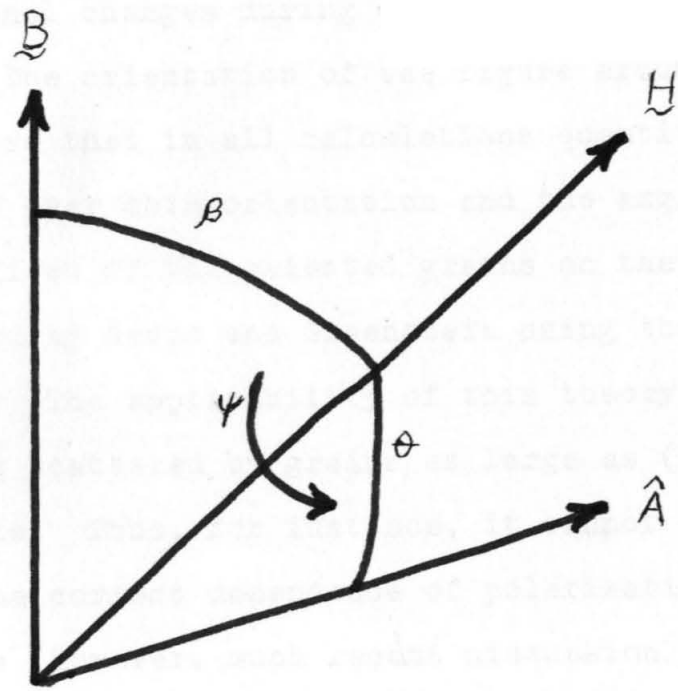

Figure 1. Definition of the orientation angles. $\underset{\sim}{B}$ is the magnetic field, $\underset{\sim}{H}$ is the angular momentum of the grain, and $\underset{\sim}{A}$ is the symmetry axis of the grain.

tates uniformly around the angular momentum, the angle between them remaining constant. That is, Θ , the angle between $\underline{H}$ and $\underline{A}$, stays constant, while Ψ, the angle between the $\underline{H}$, $\underline{B}$- plane and the $\underline{H}$, $\underline{A}$- plane, increases uniformly. The effect of the torques is to produce gradual changes in β, Θ , and the magnitude of $\underline{H}$. The use of "slow" in the beginning of this paragraph means that the fractional changes during one nutation are much less than one. The orientation of the figure around $\underline{B}$ will be random, so that in all calculations quantities will be averaged over this orientation and the angle Ψ. (9)

The effect of the oriented grains on the starlight was computed by Davis and Greenstein using the theory of Gans.(4) The applicability of this theory to visible light being scattered by grains as large as (10^{-5}) cm. is questionable. Thus, for instance, it cannot be expected to yield the correct dependence of polarization on wavelength.(10) However, much recent discussion considers smaller grains, (11) and even for the larger grains the theory gives some estimate of the effect and no really better theory is available. In any case, the results found here for the distribution of orientations could be used with any future scattering theory. Their use with the Gans theory at least gives the opportunity to compare the results of a more accurate statistical

theory with the simpler estimate of Davis and Greenstein.

The result which Davis and Greenstein find upon applying the Gans theory is that the effect of the grains can be expressed in terms of three quantities: σ_A, the extinction cross-section for a spheroid whose axis of symmetry is parallel to the electric vector of the light, σ_T, the extinction cross-section for a spheroid whose transverse axis is parallel to the electric vector, and F, a quantity giving a measure of the deviation from randomness in the orientation of the dust grains. The ratio σ_A/σ_T is given for grains of various eccentricities in table 2 of Davis and Greenstein. The definition of F is in terms of an integral, given below, over the orientation distribution function, done in such a way that for grains obeying the Gans theory, ρ/E_1 is proportional to F. Let the direction of propagation of the light make an angle $(\nu + \pi/2)$ with $\underline{B}$. Then, if all of the grains are of the same size and shape, the result is

$$\frac{\rho}{E_1} = 18.6\ F \cos^2\nu \frac{(\sigma_A/\sigma_T)-1}{(\sigma_A/\sigma_T)+2} \qquad (3)$$

If the grains are of various sizes the result must be expressed in terms of integrals over the grain sizes. However, what we might think of as the polarizing cross-

section for a single grain is proportional to the value of F for that size. Thus, for the Gans theory, σ_A and σ_T give the scattering properties of individual grains, while F contains the information necessary about the distribution of orientations.

The definition of F is given by equation (44) of Davis and Greenstein.(2) If we change variables to

$$r = \cos\beta$$
$$s = \cos\theta \tag{4}$$
$$-1 \leq r,s \leq 1$$

and let

$$p(r,s) = p_e(r,s) + p_1(r,s) \tag{5}$$

be the probability density of orientations, i.e. pdrds is the fraction of the grains in the range between r and r+dr and between s and s+ds, with p_e being the equilibrium density in the absence of a magnetic field, then the definition becomes

$$F = \int\int [r^2 s^2 + \tfrac{1}{2}(1-r^2)(1-s^2)]\, p_1(r,s)\, dr\, ds \tag{6}$$

The main results of this paper will be the calculation of p(r,s) and F. If the Gans scattering theory is used, F contains all the information necessary about the distribution of orientations. For any other scattering theory, the complete description contained in p will have to be used.

3. Paramagnetic Relaxation Mechanism

We now give the results of the analysis of the paramagnetic relaxation mechanism which are of interest here.(12) As stated before, in a time varying external field the magnetization will not exactly follow the applied field. A dust grain rotating in a constant external field can be thought of as standing still with the field rotating around it. This equivalent varying field will produce a component of magnetization perpendicular to itself. Thus, there is an apparent dragging of the magnetization away from the direction of the applied field by the motion of the grain. Analyzing the situation on this basis, Davis and Greenstein found that the important term in the magnetization perpendicular to the applied field is given by (9)

$$\underset{\sim}{M} = \frac{\chi''}{\omega}(\underset{\sim}{\omega} \times \underset{\sim}{B}) \tag{7}$$

where χ'' is the imaginary part of the complex susceptibility, and measures the angle through which $\underset{\sim}{M}$ is dragged away from $\underset{\sim}{B}$, and $\underset{\sim}{\omega}$ is the grain's angular velocity. For the expected grain compositions and internal grain temperature, T_g, they find that

$$\chi'' = 2.5(10^{-12})\frac{\omega}{T_g} \tag{8}$$

In the cases of interest, T_g is expected to be about $10^{\circ}K$.

Let V be the volume of the grain,

I be the moment of inertia about the symmetry

axis,

and γI be the moment of inertia about an axis perpendicular to the symmetry axis.

Define D, a constant for a given grain, by

$$D = \frac{\chi''}{\omega}\frac{V}{I\gamma} = \frac{2.5(10^{-12})}{T_g}\frac{V}{I\gamma} \tag{9}$$

The magnetization will produce a torque given by $V(\underset{\sim}{M}x\underset{\sim}{B})$. Then, further, this torque produces changes in the orientation variables at the rates (13).

$$\frac{d\beta}{dt} = -DB^2 \sin\beta \cos\beta (\gamma\cos^2\theta + \sin^2\theta) \tag{10}$$

and

$$\frac{d\theta}{dt} = +DB^2(\gamma-1) \sin\theta \cos\theta (1 - \frac{1}{2}\sin^2\beta) \tag{11}$$

and decreases the magnitude of the angular momentum, H, at a rate given by

$$\frac{d(H^2)}{dt} = -2DB^2H^2 \sin^2\beta (\gamma\cos^2\theta + \sin^2\theta) \tag{12}$$

For a spherical grain of density 1 gm/cm^3, internal temperature 10°K, and radius (10^{-5}) cm, $D = 6x(10^{-3})$ sec^{-1}gauss^{-2}. For a magnetic field of (10^{-5}) gauss, this gives a typical time rate of $DB^2 = 6x(10^{-13})$ sec^{-1}.

4. The Fokker-Planck Equation

The grains will also be subject to random torques due to bombardment by the atoms of the surrounding hydrogen gas. These will give random changes in the orientation variables in addition to the steady changes described by equations 10, 11, and 12. The Fokker-Plank

equation is a differential equation of the diffusion type for describing the effects on a probability distribution of such random changes of the variables. There is an extensive discussion of it, as well as other stochastic problems in physics, in an article by S. Chandrasekher. (14) The equation, as it applies to the present case, will be given here. The justification for this particular statement of it will be given in Appendix I.

Let us suppose for a moment that the set of variables whose probability distribution interests us are $x_1,\ldots,x_n$. Let $w(x_1,..,x_n,t)$ be the probability distribution at time t. That is, $Wdx_1..dx_n$ is the probability that at the time t the i^{th} variable is in the range dx_i around x_i for $i = 1,..,n$. The x_i of a representative particle will have a law of motion for small Δt such as

$$\Delta x_i = \dot{x}_i \Delta t + \delta x_i \qquad i = 1,\ldots,n \ , \tag{13}$$

where Δx_i is the change in x_i during the time interval Δt, $\dot{x}_i$ is a steady rate of change due to some known external force which may be a function of the x_i,(i.e. the rates given by equations 10, 11 and 12) and δx_i are random changes, the knowledge of which is only statistical (i.e. the effects of collisions with hydrogen gas atoms). Thus δx_i will be described by some transition probability $\Psi(\delta x_1,..,\delta x_n;x_1,..,x_n;\Delta t)\ d(\delta x_1)..d(\delta x_n)$ which gives

the probability of having changes $\delta x_1,\ldots,\delta x_n$ in $d(\delta x_1)\ldots d(\delta x_n)$ during time Δt if the current values of the variables are $x_1,\ldots,x_n$. Denote the expectation values of the δx_i and their products, i.e., the various moments of the distribution Ψ, by $E_i\Delta t$ and $E_{ij}\Delta t$. That is define E_i and E_{ij} by

$$E_i\Delta t=\int\ldots\int \delta x_i \Psi(\delta x_r;x_r;\Delta t)d(\delta x_1)..d(\delta x_n) \tag{14}$$

and

$$E_{ij}\Delta t=\int\ldots\int \delta x_i\delta x_j \Psi(\delta x_r;x_r;\Delta t)d(\delta x_1)..d(\delta x_n) \tag{15}$$

The notation anticipates that the first and second moments will be proportional to Δt in cases of interest. If, further, the third and higher moments are proportional to higher powers of Δt, the Fokker-Planck equation of the probability distribution W is

$$\frac{\partial W}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}(WE_i) + \sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j}(WE_{ij}) - \sum_i \frac{\partial}{\partial x_i}(W\dot{x}_i) \tag{16}$$

II. Formulation of the Alignment Equation

1. Plan of the calculation

The computation will then run as follows. Select a suitable set of variables, x_i; find the coefficients E_i and E_{ij} in terms of these variables for a dust grain immersed in a cloud of hydrogen gas; find the $\dot{x}_i$ from equations 10, 11, and 12; find the solution of the Fokker-Planck equation 16 for equilibrium, $\partial w/\partial t = 0$; calculate F from its definition, equation 6. Chapter IV contains a brief summary of the calculational procedures which may be useful as a guide to the rest of the paper.

The variables which the author found most convenient for formulating the Fokker-Planck equation are

$$\begin{aligned} \mu &= H \cos \theta \\ \eta &= H \cos \beta \\ \zeta &= H^2 \end{aligned} \tag{17}$$

with ranges $-\infty < \mu < \infty$ $-\infty < \eta < \infty$ $0 \leq \zeta < \infty$.

H must be included along with the angular variables since, as will be seen, the effects of grain-atom collisions depend on the grain's angular momentum in an important way. It is much easier, however, to first find the expected values of the changes in the components of the angular momentum in some cartesion coordinate system and then relate these to the changes in

μ, η, ζ by

$$\mu = \underline{H}\cdot\hat{A}$$

$$\eta = \underline{H}\cdot\underline{B}/B \qquad (18)$$

$$\zeta = \underline{H}\cdot\underline{H}$$

A collision between a hydrogen atom and a grain will take place so quickly that it can be considered to produce an impulse, $\delta\underline{H}$, of angular momentum, so that only $\underline{H}$, and therefore β and θ will be changed, but not the orientation of the grain in space. The grain's reorientation will follow from its nutational motion around the new $\underline{H}$.

For a coordinate system fixed in the grain, the expectation values of $\delta\underline{H}$ will be independent of the orientation in space of the grain and its attached coordinate system, being the same for all grains of the same size and shape. If the expectation values of components of $\delta\underline{H}$ resolved along axes fixed in space are computed, however, they will depend on the orientation of the grain. Thus it will be simplest to consider $\delta\underline{H}$ resolved into components along axes fixed in the grain and then find the effect on the orientation variables by using equations 18, which are scalar equations and true in any coordinate system.

The plan followed will be to first analyze the effect of a single hydrogen atom impact, then, from the

probability of a given type of impact, find the moments of the distribution of $\delta\underline{H}$, using a coordinate system fixed in the grain, and finally to translate these into μ, η, ζ by using equations 18. In all of this, only terms of lowest order in the ratio of hydrogen atom to grain mass will be kept.

2. Effect of a Single Collision

First let us find the effect of a single impact. Let

$\underline{b}$ be the velocity of the hydrogen atom initially,

$\underline{b}_1$ be the velocity of the hydrogen atom after the impact,

m be the mass of the hydrogen atom,

$\underline{r}$ be the point of impact on the grain's surface,

$\underline{\omega}$ be the angular velocity of the grain,

J be the inertia tensor of the dust grain, and

M be the grain's mass.

The angular momentum delivered to the grain by the impact will be that lost by the hydrogen atom. Thus

$$\delta\underline{H} = m\underline{r} \times (\underline{b} - \underline{b}_1) \qquad (19)$$

The resulting atom velocity, $\underline{b}_1$, will be a function of $\underline{b}$, $\underline{\omega}$ and the model chosen for the collision. If it is supposed that the atom hits, sticks, and then is thrown off with the local velocity of the grain surface, $\underline{b}_1 = \underline{\omega} \times \underline{r}$, and

$$\delta \underline{H} = m\underline{r} \times (\underline{b} - \underline{\omega} x \underline{r}) \qquad (20)$$

For a second model suppose that the atom makes an elastic collision and that the outgoing velocity is parallel to the incoming velocity in the coordinate system in which the local grain surface is stationary. In this latter coordinate system, in the lowest order of atom to grain mass, the atom will be making an elastic collision with an infinitely heavy target, and so will merely reverse its velocity. In this coordinate system the incoming velocity will be $\underline{b} - (\underline{\omega} \times \underline{r})$. When this is reversed and transformed back to the original coordinate system, it gives for the outgoing velocity $\underline{b}_1 = - (\underline{b} - \underline{\omega} \times \underline{r}) + \underline{\omega} \times \underline{r} = - \underline{b} + 2(\underline{\omega} \times \underline{r})$. Thus, this model gives

$$\delta \underline{H} = m\underline{r} \times (\underline{b}-\underline{b}_1) = 2m\underline{r} \times (\underline{b}-\underline{\omega}x\underline{r}) \qquad (21)$$

The difference between quotations 20 and 21 is only in the factor of 2. Various other reasonable models give the same results, i.e. expression (21) with a different numerical factor. This can be taken into account by always using equation 21, and assigning to the hydrogen atom an effective mass, m^*, which depends on the collision model chosen. We will therefore use

$$\delta \underline{H} = 2m^* \underline{r} \times [\underline{b} - (\underline{\omega}x\underline{r})] \qquad (21^*)$$

where $m^*=m$ for an elastic collision and $m^*= m/2$ for a completely inelastic collision.

3. Calculation of Moments of $\delta\underline{H}$

The next step in finding the diffusion coefficients in the Fokker-Planck equation is to calculate the moments of the distribution of $\delta\underline{H}$ from the collisions. Let $Nd\underline{b}dSdt$ be the number of collisions on an element, dS, of the grain's surface during a time dt due to hydrogen atoms in a range of velocities $d\underline{b}=db_x db_y db_z$ around the velocity $\underline{b}$. Further, suppose that dt is much less than the average time between collisions on a given grain. Then $Nd\underline{b}dSdt$ will also be the probability of having a collision of the type indicated during time dt. Note that the overwhelming probability is to have no collision. As discussed in Appendix I, the moments of $\delta\underline{H}$ calculated on the distribution for a time $\Delta t \gg$ dt during which many collisions can be expected to occur give the same results for E_i and E_{ij} as a calculation based on the distribution for the time dt. That is

$$E_i dt = dt\int \delta H_i N d\underline{b} dS \qquad i=x,y,z \qquad (22)$$

and

$$E_{ij} dt = dt\int (\delta H_i)(\delta H_j) N d\underline{b} dS \qquad i,j=x,y,z \qquad (23)$$

where the integrations are taken over all of the grain surface and atom velocities.

The velocity of an atom relative to a point on the grain surface is $\underline{b} - (\underline{\omega} \times \underline{r})$. For a collision to occur, the normal component of this velocity must be toward the grain. This, combined with the assumption of a Maxwell

distribution of atom velocities, gives

$$N d\underset{\sim}{b} ds dt = \begin{cases} -Ae^{-mb^2/2kT}\, \underset{\sim}{n}\cdot[\underset{\sim}{b}-(\underset{\sim}{\omega}x\underset{\sim}{r})]\, d\underset{\sim}{b} ds dt \\ \qquad\qquad \text{for } \underset{\sim}{n}\cdot[\underset{\sim}{b}-(\underset{\sim}{\omega}x\underset{\sim}{r})] \le 0 \\ 0 \quad \text{for } \underset{\sim}{n}\cdot[\underset{\sim}{b}-(\underset{\sim}{\omega}x\underset{\sim}{r})] \ge 0 \end{cases} \tag{24}$$

Here $\underset{\sim}{n}$ is a unit vector drawn <u>outward</u> normal to the grain's surface, and A is a constant to normalize the hydrogen atom density to n_H atoms per cm^3. Thus

$$n_H = \int Ae^{-mb^2/2KT} d\underset{\sim}{b}$$

or

$$A = n_H\left(\frac{m}{2\pi kT}\right)^{3/2} = \pi^{-3/2} n_H c^{-3} \tag{25}$$

where in the last form we have introduced the hydrogen atom characteristic velocity, given by

$$c = \left(\frac{2kT}{m}\right)^{1/2} \tag{26}$$

To simplify some of the calculations, let us denote the vector composed of the three first moments by

$$E_{\underset{\sim}{H}} = (E_x, E_y, E_z) \tag{27}$$

and the symmetric matrix of the six second moments by

$$E_{\underset{\sim}{\underset{\sim}{HH}}} = \begin{pmatrix} E_{xx} & E_{xy} & E_{xz} \\ E_{yx} & E_{yy} & E_{yz} \\ E_{zx} & E_{zy} & E_{zz} \end{pmatrix} \tag{28}$$

Also let $(\delta\underset{\sim}{H})(\delta\underset{\sim}{H})$ be either the dyadic or its matrix representation. The x, y, z coordinates, as implied before, refer to a system fixed with respect to the grain, not with respect to space. We have not yet

specified how it is oriented in the grain.

The definitions of equations 27 and 28 then give

$$E_{\underline{H}} = \int N(\delta \underline{H})\, d\underline{b}\, ds \tag{29}$$

and

$$E_{\underline{HH}} = \int N(\delta \underline{H})(\delta \underline{H})\, d\underline{b} dS \tag{30}$$

Equations 21 and 24 give $\delta \underline{H}$ and N. These must be substituted in equations 29 and 30 and the indicated five-fold integrations must be carried out. The results, to lowest order in the ratio of atom to grain mass, turn out to depend on a simple matrix whose elements are certain integrals over the surface of the grain and on the inertia tensor of the grain. (See equations 42 and 43 below.) The result holds for an arbitrary grain shape, and can then be specialized to the spheroids being considered here.

When we estimate the order of magnitude of various terms in the integrals, a term involving $\underline{\omega}$ and $\underline{r}$ can in general be estimated to have the magnitude $\omega r/c$, c being the characteristic velocity of the atoms. (see equation 26) For orders of magnitude, $kT \sim \frac{1}{2} I \omega^2 \sim \frac{1}{2} M r^2 \omega^2$, M being the mass of the grain. Thus we have

$$\frac{\omega r}{c} \sim \left(\frac{2kT}{M}\right)^{1/2} \Big/ \left(\frac{2kT}{m}\right)^{1/2} \sim \left(\frac{m}{M}\right)^{1/2} \tag{31}$$

and only terms of lowest order in this ratio are to be kept.

To handle the limits on the velocity integrations implied by equation 24, introduce the velocity $\underset{\sim}{S}$ given by

$$\underset{\sim}{b} = c\underset{\sim}{s} \tag{32}$$

At each point on the surface of the grain attach a coordinate system for $\underset{\sim}{S}$, as shown in figure 2, with components u, v, w oriented so that the u-axis is along $\underset{\sim}{n}$, the v-axis is normal to $\underset{\sim}{n}$ and to the z-axis, i.e. is parallel to the xy-plane, and the w-axis is in the $\underset{\sim}{n}$, z-plane. With this coordinate system, $\underset{\sim}{n}\cdot\underset{\sim}{b} = cu$, so that the region of integration in velocity space implied by equation 24 is

$$-\infty \leq v, w \leq \infty$$

$$-\infty \leq u \leq \frac{\underset{\sim}{n}\cdot(\underset{\sim}{\omega}x\underset{\sim}{r})}{c}$$

Further, we will break up the u integration into two parts, first integrating for $-\infty \leq u < 0$ and then adding the integral for u between 0^- and $\underset{\sim}{n}\cdot(\underset{\sim}{\omega}x\underset{\sim}{r})/c$.

If we substitute equation 21 and 24 in 29, using the $\underset{\sim}{s}$ coordinate system, we get

$$-\frac{\pi^{3/2}}{2n_H m^* c^2}\,\underset{\sim}{E}_H = \iint_{surf} \int_{v,w=-\infty}^{\infty} \int_{u=-\infty}^{0} (\ I\ )\, du\,dv\,dw\,dS + \iint_{surf} \int_{v,w=-\infty}^{\infty} \int_{u=0}^{\underset{\sim}{n}\cdot\underset{\sim}{\omega}x\underset{\sim}{r}/c} (\ I\ )\, du\,dv\,dw\,dS \tag{33}$$

The integrand in both integrals is the same, and is

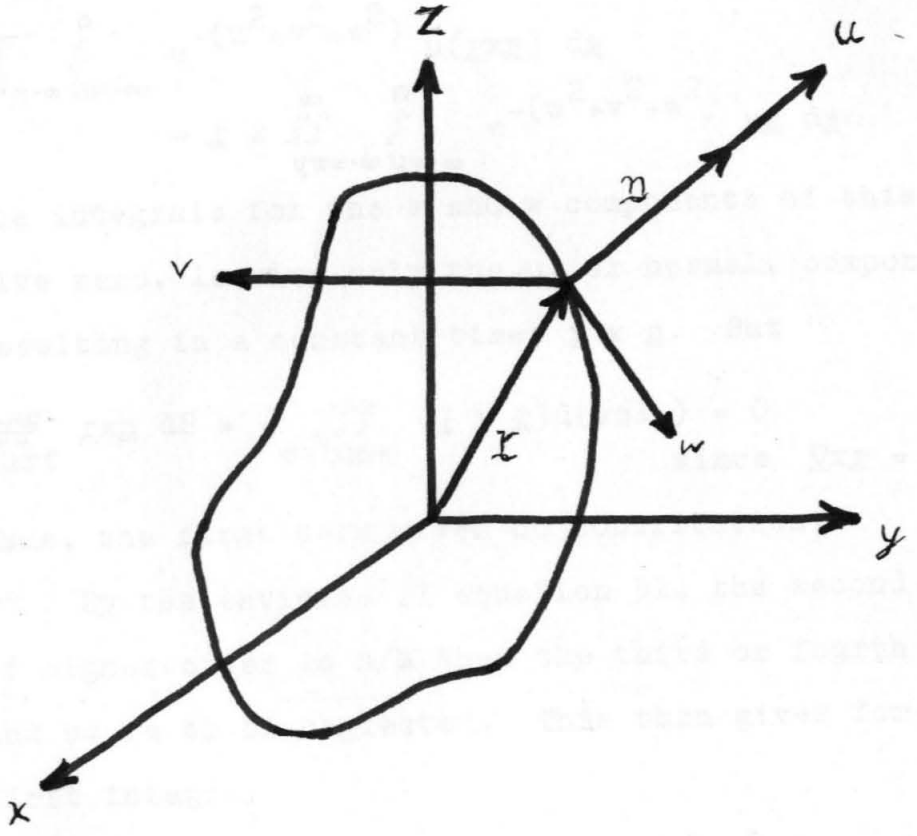

Figure 2. Placement of $\underline{s}$ space coordinate system for velocity integrations.

given by

$$(\text{I}) = \left[\underset{\sim}{r}\times\underset{\sim}{s} - \frac{\underset{\sim}{r}\times(\underset{\sim}{\omega}\times\underset{\sim}{r})}{c}\right]\left[u - \frac{\underset{\sim}{n}\cdot(\underset{\sim}{\omega}\times\underset{\sim}{r})}{c}\right] e^{-(u^2+v^2+w^2)} \qquad (34)$$

$$= \left[u(\underset{\sim}{r}\times\underset{\sim}{s}) + \frac{\underset{\sim}{r}\times(\underset{\sim}{\omega}\times\underset{\sim}{r})}{c}\,\frac{\underset{\sim}{n}\cdot(\underset{\sim}{\omega}\times\underset{\sim}{r})}{c} - (\underset{\sim}{r}\times\underset{\sim}{s})\frac{\underset{\sim}{n}\cdot(\underset{\sim}{\omega}\times\underset{\sim}{r})}{c} - \frac{u\,\underset{\sim}{r}\times(\underset{\sim}{\omega}\times\underset{\sim}{r})}{c}\right] e^{-(u^2+v^2+w^2)}$$

Consider the first integral of equation 33. The velocity space integral of the first term is

$$\iint_{v,w=-\infty}^{\infty} \int_{u=-\infty}^{0} e^{-(u^2+v^2+w^2)}\, u(\underset{\sim}{r}\times\underset{\sim}{s})\, d\underset{\sim}{s}$$

$$= \underset{\sim}{r} \times \iint_{v,w=-\infty}^{\infty} \int_{u=-\infty}^{0} e^{-(u^2+v^2+w^2)}\, u\underset{\sim}{s}\, d\underset{\sim}{s}$$

The integrals for the v and w components of this will give zero, leaving only the u, or normal, component, resulting in a constant times $\underset{\sim}{r} \times \underset{\sim}{n}$. But

$$\oiint_{\text{surf}} \underset{\sim}{r}\times\underset{\sim}{n}\, dS = -\iiint_{\text{volume}} (\nabla \times \underset{\sim}{r})\, d(\text{vol.}) = 0$$

since $\nabla\times\underset{\sim}{r} = 0$.

Thus, the first term gives no contribution.

By the estimate of equation 31, the second term is of higher order in m/M than the third or fourth terms and so is to be neglected. This then gives for the first integral

$$-\oiint_{\text{surf}} \left\{ \frac{\underset{\sim}{n}\cdot(\underset{\sim}{\omega}\times\underset{\sim}{r})}{c}\, \underset{\sim}{r}\times\underset{\sim}{n} \iint_{v,w=-\infty}^{\infty} \int_{u=-\infty}^{0} u e^{-(u^2+v^2+w^2)}\, dudvdw + \frac{\underset{\sim}{r}\times(\underset{\sim}{\omega}\times\underset{\sim}{r})}{c} \iint_{v,w=-\infty}^{\infty} \int_{u=-\infty}^{0} u e^{-(u^2+v^2+w^2)}\, dudvdw \right\} dS \qquad (35)$$

$$= \frac{\pi}{2c} \oiint_{\text{surf}} \left\{ (\underset{\sim}{r}\times\underset{\sim}{n})(\underset{\sim}{r}\times\underset{\sim}{n})\cdot\underset{\sim}{\omega} + \underset{\sim}{r}\times(\underset{\sim}{\omega}\times\underset{\sim}{r}) \right\} dS$$

Now consider the second integral of equation 33. In it, the maximum value of $|u|$ is $|\underline{n}\cdot(\underline{\omega}x\underline{r})/c|$ which is of the order $(m/M)^{1/2} \ll 1$ by equation 31. Thus, for this integral, $e^{-u^2} = 1$ plus second order terms. The four terms in the integrand will once again be those given by equation 34. When integrated over v and w, they become a constant times

$$\left[u^2(\underline{r}x\underline{n}) + \frac{\underline{r}x(\underline{\omega}x\underline{r})}{c}\,\frac{\underline{n}\cdot(\underline{\omega}x\underline{r})}{c} - u(\underline{r}x\underline{n})\frac{\underline{n}\cdot(\underline{\omega}x\underline{r})}{c} - u\frac{\underline{r}x(\underline{\omega}x\underline{r})}{c} \right]$$

Since u is limited to $\underline{n}\cdot(\underline{\omega}x\underline{r})/c$, all of these terms will be of second order in $\omega r/c$ in the range of integration. The integration

$$\int_{u=0}^{\underline{n}\cdot(\underline{\omega}x\underline{r})/c} [\ ]\ du$$

then makes them contribute only in third order in $\omega r/c$, so that they are completely negligible, and $-\pi^{3/2}E_{\underline{H}}/\ n_H m^* c^2$ is given by equation 35.

To evaluate $E_{\underline{HH}}$, we must substitute equations 21 and 24 in equation 30. The regions of integration will be the same as those for $E_{\underline{H}}$ in equation 33, but the integrand will be different. The integrand for

$-\pi^{3/2}E_{\underline{HH}}/4n_H(m^*)^2c^3$ will be

$$e^{-(u^2+v^2+w^2)} \left\{ (\underline{r}x\underline{s})(\underline{r}x\underline{s}) + \frac{[\underline{r}x(\underline{\omega}x\underline{r})][\underline{r}x(\underline{\omega}x\underline{r})]}{c^2} - \frac{(\underline{r}x\underline{s})[\underline{r}x(\underline{\omega}x\underline{r})]}{c} \right. \tag{36}$$

$$\left. -[\underline{r}x(\underline{\omega}x\underline{r})](\underline{r}x\underline{s}) \right\} \left\{ u - \frac{\underline{\omega}\cdot(\underline{r}x\underline{n})}{c} \right\}$$

Each term in the first set of curly brackets is a dyadic. The term of lowest order will be the dyadic $u(\underset{\sim}{r}x\underset{\sim}{s})(\underset{\sim}{r}x\underset{\sim}{s})\exp\left[-(u^2+v^2+w^2)\right]$. Any terms which are odd functions of v or w in the expansion of this dyadic will integrate to zero. The matrix representing this dyadic, written out in the u, v, w coordinate system, and leaving out terms odd in v or w, is

$$ue^{-(u^2+v^2+w^2)}\begin{pmatrix}(r_w^2v^2+r_v^2w^2) & -r_ur_vw^2 & -r_ur_wv^2\\ -r_vr_uw^2 & (r_u^2w^2+r_w^2u^2) & -r_vr_wu^2\\ -r_wr_uv^2 & -r_wr_vu^2 & (r_v^2u^2+r_u^2v^2)\end{pmatrix} \tag{37}$$

where r_u, r_v, and r_w are the components of $\underset{\sim}{r}$ along the u, v, w axes. Note that the matrix is symmetric. If this is integrated over the range of velocity variables in the first integral of equation 33, the result is

$$-\frac{\pi}{4}\begin{pmatrix}(r_v^2+r_w^2) & -r_ur_v & -r_ur_w\\ -r_vr_u & (2r_w^2+r_u^2) & -2r_vr_w\\ -r_wr_u & -2r_wr_v & (2r_v^2+r_u^2)\end{pmatrix} \tag{38}$$

For the same reasons as used in analyzing $E_{\underset{\sim}{H}}$, the integral over the second region of equation 33 will contribute only higher order terms, so that $-\pi^{3/2}E_{\underset{\sim}{H}\underset{\sim}{H}}/4\pi_H(m^*)^2c^3$ will be given by the integral over the surface of the grain of equation 38.

To do this surface integral, we must first transform equation 38, which is written in terms of a coordinate system whose orientation depends upon the location of the element of surface of the grain, to a coordinate system which is fixed in direction with respect to the grain. This can be done most easily by putting equation 38 in a form which contains no references to a coordinate system at all. Direct expansion in the u, v, w coordinate system will show that equation is equal to the matrix representing the dyadic

$$-\frac{\pi}{4}\left[r^2 U + (\underline{r}x\underline{n})(\underline{r}x\underline{n}) - (\underline{r})(\underline{r})\right]$$

where U is the unit dyadic. This finally gives

$$-\frac{\pi^{3/2}}{4n_H(m^*)^2c^3}\, E_{\underline{\underline{HH}}} = -\frac{\pi}{4}\iint_{\text{surf}} \left[r^2 U + (\underline{r}x\underline{n})(\underline{r}x\underline{n}) - (\underline{r})(\underline{r})\right] d(\text{surf.}) \qquad (39)$$

The results for $E_{\underline{H}}$ and $E_{\underline{\underline{HH}}}$, equations 35 and 39, can be unified. Consider the integrand in equation 35. If $\underline{r}\ x(\underline{\omega}x\underline{r})$ is expanded, the integrand can be written as $(\underline{r}x\underline{n})(\underline{r}x\underline{n})\cdot\underline{\omega} + r^2\underline{\omega} - (\underline{r})(\underline{r}\cdot\underline{\omega})$ which equals $\left[(\underline{r}x\underline{n})(\underline{r}x\underline{n}) + r^2 U - (\underline{r})(\underline{r})\right]\cdot\underline{\omega}$.

Let J^{-1} be the inverse of the inertia tensor of the grain, so that we can write the angular momentum by the relation $\underline{\omega} = J^{-1}\underline{H}$. Define the dyadic, or its matrix representation, G, by

$$G = \oint\!\!\oint_{surf} [r^2 U - (\underline{r})(\underline{r}) + (\underline{r}\times\underline{n})(\underline{r}\times\underline{n})]\, d(surf) \tag{40}$$

and the constant $$g = \pi^{-1/2} n_H m^* c \tag{41}$$

Then

$$\underline{E}_{\underline{H}} = - gGJ^{-1} \underline{H} \tag{42}$$

and $$E_{\underline{HH}} = gm\, c^2\, G \tag{43}$$

The results so far are valid for arbitrarily shaped grains. They must now be specialized to the spheroids that are of interest here. Take the axis of symmetry of the grain as the z-axis. It can easily be seen that the matrix G will be diagonal in this coordinate system. For consider the definition of G, equation 40. The first two terms, $\oint\!\!\oint_{surf} [r^2 U - (\underline{r})(\underline{r})]\, dS$ are just the inertia tensor for a hollow grain of unit surface density of mass, and this coordinate system is certainly a principle axis system for such a body. For the third term, $(\underline{r}\times\underline{n})$ will be a vector parallel to the x, y-plane, tangent to the grain surface, and of constant magnitude for a given value of z, so that the integral of the off diagonal components of $(\underline{r}\times\underline{n})(\underline{r}\times\underline{n})$ around a zone of constant z will be zero. Thus in the coordinate system with z-axis along the axis of rotational symmetry of the grain,

$$G = \begin{pmatrix} \alpha h & 0 & 0 \\ 0 & \alpha h & 0 \\ 0 & 0 & h \end{pmatrix} \tag{44}$$

and is determined by the two constants α and h.

This coordinate system will also be a principal axis system for the tensor of inertia of the actual grain, which may be written

$$J = \begin{pmatrix} \gamma I & 0 & 0 \\ 0 & \gamma I & 0 \\ 0 & 0 & I \end{pmatrix}$$ and its inverse is

$$J^{-1} = \begin{pmatrix} 1/\gamma I & 0 & 0 \\ 0 & 1/\gamma I & 0 \\ 0 & 0 & 1/I \end{pmatrix} \tag{45}$$

Putting equations 44 and 45 into equations 42 and 43 then gives

$$\begin{aligned} E_x &= - g\frac{h}{I}\frac{\alpha}{\gamma} H_x \\ E_y &= - g\frac{h}{I}\frac{\alpha}{\gamma} H_y \\ E_z &= - g\frac{h}{I} H_z \end{aligned} \tag{46}$$

and

$$\begin{aligned} E_{xx} &= gm^* c^2 h\alpha \\ E_{yy} &= gm^* c^2 h\alpha \\ E_{zz} &= gm^* c^2 h \\ E_{xy} = E_{xz} = E_{yz} &= 0 \end{aligned} \tag{47}$$

These results, equations 46 and 47 hold for any axially symmetric grain. The surface integrals of equation 40 must now be carried out for the particular

grain shape under consideration. For a spheroid with diameter along the rotation axis 2b and perpendicular diameter 2a, some elementary integrations give

$$h=\pi a^4\left\{1+\frac{\varepsilon^2}{(\varepsilon^2-1)2}+\left[2-\frac{\varepsilon^2}{2(\varepsilon^2-1)}\right]\frac{\varepsilon^2}{|\varepsilon^2-1|^{1/2}}\times \left(\begin{matrix}\sin^{-1}\\ \sinh^{-1}\end{matrix}-1\right)\left(\frac{|\varepsilon^2-1|^{1/2}}{\varepsilon}\right)\right\} \tag{48}$$

$$\alpha h=\pi a^4\left\{1+\varepsilon^2-\frac{\varepsilon^4}{4(\varepsilon^2-1)}+\left[2+\frac{\varepsilon^2}{\varepsilon^2-1}\right]\frac{\varepsilon^4}{|\varepsilon^2-1|^{1/2}\,4}\times \left(\begin{matrix}\sin^{-1}\\ \sinh^{-1}\end{matrix}-1\right)\left(\frac{|\varepsilon^2-1|^{1/2}}{\varepsilon}\right)\right\} \tag{49}$$

Here $\varepsilon=b/a$, and $\sinh^{-1}(\ )$ is to be used for an oblate spheroid ($a>b$), $\sin^{-1}(\ )$ for a prolate spheroid ($a<b$). The moments of inertia can easily be found if it is supposed that the spheroid is of uniform density. They are

$$I = M\,\frac{2}{5}\,a^2 \tag{50}$$

$$\gamma I = M\,\frac{a^2+b^2}{5}$$

For the case of dust grains which are nearly spheres, b/a will be approximately one. If we expand the expressions for h and αh, equation 48 and 49, in powers of $(b/a)^2-1$ up to first order we get

$$\begin{aligned} h &\doteq \pi a^4\left[\frac{8}{3}+\frac{16}{15}\left[(b/a)^2-1\right]\right] \\ \alpha h &\doteq \pi a^4\left[\frac{8}{3}+\frac{32}{15}\left[(b/a)^2-1\right]\right] \end{aligned} \tag{51}$$

and

$$\alpha-1 \doteq \frac{2}{5}\left[(b/a)^2-1\right] \tag{52}$$

An expansion to first order, using equation 50 will give

$$\gamma-1 = \frac{1}{2}\left[(b/a)^2-1\right] \tag{53}$$

Thus, the relation between the two parameters which account for the non-sphericity of the grain, for grains of uniform density, is

$$(\alpha-1) \doteq \frac{4}{5}\,(\gamma-1) \tag{54}$$

up to an error of second order.

4. Calculation of Terms in the Fokker-Planck Equation

Equations 46 and 47 give the effects of the collisions on the value of $\underline{H}$ relative to a coordinate system fixed in the grain. These must be related to the variables describing the orientation of the grain by use of equation 18,

$$\begin{aligned} \mu &= \underline{H}\cdot\hat{A} \\ \nu &= \underline{H}\cdot\hat{B} \\ \zeta &= \underline{H}\cdot\underline{H} \end{aligned} \tag{18}$$

where now $\hat{B}$ is a unit vector in the direction of $\underline{B}$. This can easily be done, since in a collision $\hat{A}$ and $\hat{B}$ remain fixed, only $\underline{H}$ changing, so that

$$\delta\mu = \hat{A}_x \delta H_x + \hat{A}_y \delta H_y + \hat{A}_z \delta H_z$$
$$\delta\nu = \hat{B}_x \delta H_x + \hat{B}_y \delta H_y + \hat{B}_z \delta H_z$$
$$\delta\zeta = (\underline{H}+\delta\underline{H})^2 - H^2 = 2[H_x \delta H_x + H_y \delta H_y + H_z \delta H_z] + (\delta H_x)^2 + (\delta H_y)^2 + (\delta H_z)^2 \qquad (55)$$

Thus what remains to be done is to express the components of $\underline{H}$, $\hat{B}$, and $\hat{A}$ in terms of μ, ν, and ζ.

To do this we must further specify the orientation of the x, y, z coordinate system. So far, it has been chosen so that the z-axis is along A, the symmetry axis of the grain. Now fix the orientation around the symmetry axis so that the y-axis is in the $\underline{H}$, $\hat{A}$-plane. (See figure 3). To fit the direction of $\hat{A}$ we must also give Ψ, the angle between the $\underline{H}$, $\hat{B}$ and $\underline{H}$, $\hat{A}$ planes. This is the nutational angle, and, as previously discussed, all effects are to be averaged over this angle.

In this coordinate system,

$$\hat{A}_x = \hat{A}_y = 0 \qquad \hat{A}_z = 1 \qquad (56)$$

$$H_x = 0 \qquad H_y = -H \sin\theta \qquad H_z = H\cos\theta = \mu \qquad (57)$$

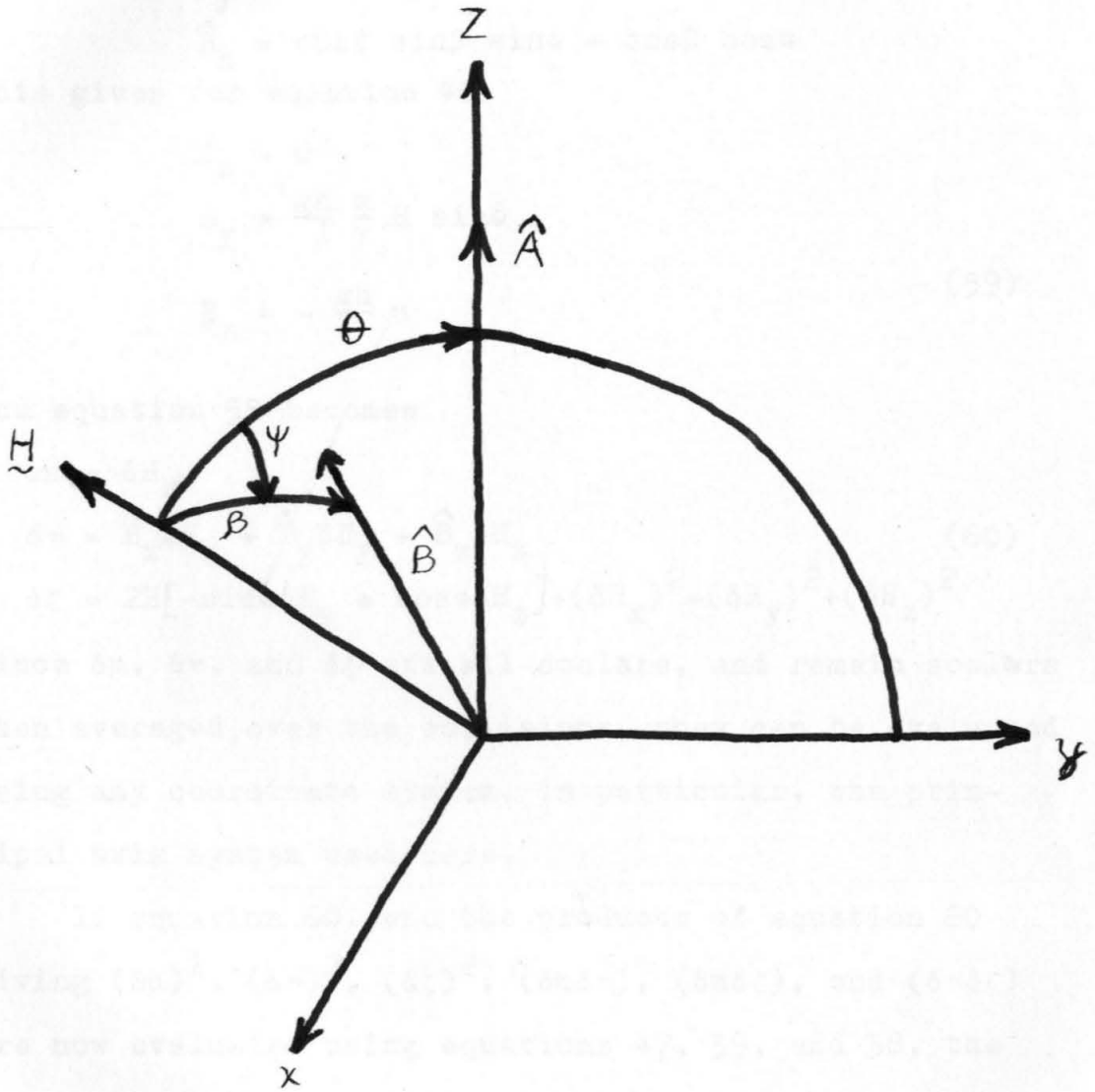

Figure 3. Orientation of the x,y,z coordinate system.

and

$$\begin{aligned} \hat{B}_x &= \sin\Psi \sin\beta \\ \hat{B}_y &= \cos\Psi \sin\beta \cos\theta - \cos\beta \sin\theta \\ \hat{B}_z &= \cos\Psi \sin\beta \sin\theta + \cos\beta \cos\theta \end{aligned} \tag{58}$$

This gives for equation 46

$$\begin{aligned} E_x &= 0 \\ E_y &= \frac{gh}{I} \frac{\alpha}{\gamma} H \sin\theta \\ E_z &= -\frac{gh}{I} \mu \end{aligned} \tag{59}$$

and equation 55 becomes

$$\begin{aligned} \delta\mu &= \delta H_z \\ \delta\eta &= \hat{B}_x \delta H_x + \hat{B}_y \delta H_y + \hat{B}_z \delta H_z \\ \delta\zeta &= 2H\left[-\sin\theta \delta H_y + \cos\theta \delta H_z\right] + (\delta H_x)^2 + (\delta H_y)^2 + (\delta H_z)^2 \end{aligned} \tag{60}$$

Since $\delta\mu$, $\delta\eta$, and $\delta\zeta$ are all scalars, and remain scalars when averaged over the collisions, they can be evaluated using any coordinate system, in particular, the principal axis system used here.

If equation 60, and the products of equation 60 giving $(\delta\mu)^2$, $(\delta\eta)^2$, $(\delta\zeta)^2$, $(\delta\mu\delta\eta)$, $(\delta\mu\delta\zeta)$, and $(\delta\eta\delta\zeta)$ are now evaluated using equations 47, 59, and 58, the average over Ψ performed, and then rewritten using the definitions of μ, η, and ζ given by equation 17, the results will be

$$\begin{aligned} E_\mu &= -\frac{gh}{I} \mu \\ E_\eta &= -\frac{gh}{I} \eta\left[\frac{\alpha}{\gamma} - \frac{\alpha\mu}{\gamma\zeta}^2 + \frac{\mu}{\zeta}^2\right] \end{aligned}$$

$$E_\zeta = \frac{gh}{I}\left[m^*c^2 I(1+2\alpha) - \frac{2\mu^2(\gamma-\alpha) + 2\alpha\zeta}{\gamma}\right]$$

$$E_{\mu\mu} = ghm^*c^2$$

$$E_{\eta\eta} = ghm^*c^2\left[\frac{(\alpha+1)}{2} + \frac{(\alpha-1)}{2\zeta}(\eta^2+\mu^2 - \frac{3\eta^2\mu^2}{\zeta})\right]$$

$$E_{\zeta\zeta} = 4ghm^*c^2\left[\alpha\zeta + (1-\alpha)\mu^2\right] \qquad (61)$$

$$E_{\mu\eta} = ghm^*c^2\,\frac{\mu\eta}{\zeta}$$

$$E_{\mu\zeta} = 2ghm^*c^2\mu$$

$$E_{\eta\zeta} = 2ghm^*c^2\left[\alpha\eta + (1-\alpha)\frac{\eta\mu^2}{\zeta}\right]$$

In these results, equation 61, all traces of the special x, y, z coordinate system have now vanished, and the moments are in their final form ready for use in the Fokker-Planck equation, 16

$$\frac{\partial W}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}(WE_i) + \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}(WE_{ij}) - \sum_i \frac{\partial}{\partial x_i}(W\dot{x}_i) \qquad (16)$$

The x_i's appearing in equation 16 must now be identified with the variables $\dot{\mu}$, $\dot{\eta}$, and $\dot{\zeta}$. It still remains, however, to express the μ, η, and ζ in the last term of equation 16 in proper form. Equations 10, 11, and 12 give the rates of change of β, θ, and H^2, and these can be related to $\dot{\mu}$, $\dot{\eta}$, and $\dot{\zeta}$ by equation 17 and the relation $\dot{H} = (1/2H)\, d/_{dt}(H^2)$. Thus

$$\dot{\mu} = \dot{H}\cos\theta - H\dot{\theta}\sin\theta = DB^2\left\{\frac{3}{2}(\gamma-1)\frac{\eta^2\mu^3}{\zeta^2} - \frac{1}{2}(\gamma-1)\frac{\mu^3}{\zeta}\right.$$
$$\left. +\left[1-\frac{1}{2}(\gamma-1)\right]\frac{\eta^2\mu}{\zeta} - \left[1+\frac{1}{2}(\gamma-1)\right]\mu\right\}$$
$$\dot{\eta} = \dot{H}\cos\beta - H\dot{\beta}\sin\beta = 0 \qquad (62)$$
$$\dot{\zeta} = (\dot{H^2}) = DB^2\left\{-2(\gamma-1)\mu^2 - 2\zeta + 2(\gamma-1)\frac{\eta^2\mu^2}{\zeta} + 2\eta^2\right\}$$

The middle one of these equations, $\dot{\eta}=0$, is to be expected, since the torque produced by the Davis-Greenstein mechanism is such as to leave the projection of $\underline{H}$ on $\underline{B}$, which is just η, constant.[15]

We can write equation 16 in the form

$$\frac{\partial W}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}\left[WE_i - \frac{1}{2}\sum_j \frac{\partial}{\partial x_j}(WE_{ij}) + W\dot{x}_i\right]$$

which has the obvious interpretation as an equation of continuity for a flow of probability density W, with the terms in brackets being the components of current. Thus $WE_i - \frac{1}{2}\Sigma_j\, \partial/\partial x_j(WE_{ij})$ is the i^{th} component of current caused by collisions and Wx_i is the i^{th} component caused by the magnetic field. Letting the three components of the current be denoted by J_η, J_μ, and J_ζ, we find that using equation 61 and equation 62 and performing the indicated differentiations gives

$$J_\mu = -ghm^*c^2\left\{\left(\frac{\mu}{m^*c^2 I} + \frac{\mu}{2\zeta}\right)W + \frac{1}{2}W_\mu + \frac{\mu\eta}{2\zeta}W_\eta + \mu W_\zeta\right\}$$
$$-DB^2 W\left\{-\frac{3}{2}(\gamma-1)\frac{\eta^2\mu^3}{\zeta^2} + \frac{1}{2}(\gamma-1)\frac{\mu^3}{\zeta}\right. \qquad (63)$$
$$\left. -\left[1-\frac{1}{2}(\gamma-1)\right]\frac{\eta^2\mu}{\zeta} + \left[1+\frac{1}{2}(\gamma-1)\right]\mu\right\}$$

$$J_\eta = -ghm^* c^2 \Big\{ \Big[\frac{(1-\alpha)\eta\mu^2}{2\zeta^2} + \frac{\alpha\eta}{2\zeta} + \frac{\eta\alpha}{Im^* c^2 \gamma} - \frac{\eta\mu^2\alpha}{Im^* c^2 \zeta\gamma} \tag{64}$$

$$+\frac{\eta\mu^2}{Im\ c^2\zeta}\Big] W + \frac{\mu\eta}{2\zeta} W_\mu + \Big[\frac{1}{4}(\alpha+1) + \frac{(\alpha-1)}{4}\Big(\frac{\eta^2}{\zeta} + \frac{\mu^2}{\zeta}$$

$$-\frac{3\eta^2\mu^2}{\zeta^2}\Big)\Big] W_\eta + \Big[\alpha\eta + (1-\alpha)\frac{\eta\mu^2}{\zeta}\Big] W_\zeta \Big\}$$

$$J_\zeta = -ghm^* c^2 \Big\{ \Big[\alpha + (1-\alpha)\frac{\mu^2}{\zeta} + \frac{2\mu^2(\gamma-\alpha)+2\alpha\zeta}{m^* c^2 \gamma I} \Big] W + \mu W_\mu \tag{65}$$

$$+\Big[\alpha\eta + (1-\alpha)\frac{\mu^2\eta}{\zeta}\Big] W_\eta + 2\Big[\alpha\zeta + (1-\alpha)\mu^2\Big] W_\zeta$$

$$-DB^2 W \Big\{ 2(\gamma-1)\mu^2 + 2\zeta - 2(\gamma-1)\frac{\eta^2\mu^2}{\zeta} - 2\eta^2 \Big\}$$

A subscript on W indicates a partial derivative with respect to that variable.

The equation of the aligning process can now be stated simply as

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial \mu} J_\mu - \frac{\partial}{\partial \eta} J_\eta - \frac{\partial}{\partial \zeta} J_\zeta \tag{66}$$

5. Conservation of Total Probability.

Since W is a probability density, $(\partial/\partial t) \int W d\mu d\eta d\zeta$ must vanish for the integral taken over the entire range of the variables. By using equation 66 and Gauss' theorem, it follows that this will be true for any W if the normal component of the current, $(J\mu, J\eta, J\zeta)$, vanishes on the boundary. From equation 17, we see that the range of the variables is

$$\mu^2 \leq \zeta \tag{67}$$

$$\eta^2 \leq \zeta$$

This will be the inside of the volume formed by the intersection of the two parabolic cylinders, $\mu^2=\zeta$ and $\eta^2=\zeta$. Their projections on the η, ζ and μ, ζ planes are shown in figure 4. The boundary $\eta^2=\zeta$ is called I and $\mu^2=\zeta$ is called II. The slope of surface I is 2η. The slope of the normal vector will be $-1/2\eta$, so that one normal vector has components $n_\eta=2\eta$, $n_\zeta=-1$, $n_\mu=0$, and the normal component of the current will be proportional to

$$J_{In} = 2\eta J_\eta - J_\zeta \tag{68}$$

The situation on surface II is exactly similar, so that

$$J_{IIn} = 2\mu J_\mu - J_\zeta \tag{69}$$

Equation 68 evaluated from equation 64 and equation 65 with $\eta^2=\zeta$, and equation 69 evaluated from equation 63 and equation 65 with $\mu^2=\zeta$ will both be found to vanish, so that the necessary condition of conservation of probability holds.

6. Maxwell-Boltzmann Solution

Another condition which the equation of the process should satisfy is that the Maxwell-Boltzmann distribution for an appropriate temperature should be an

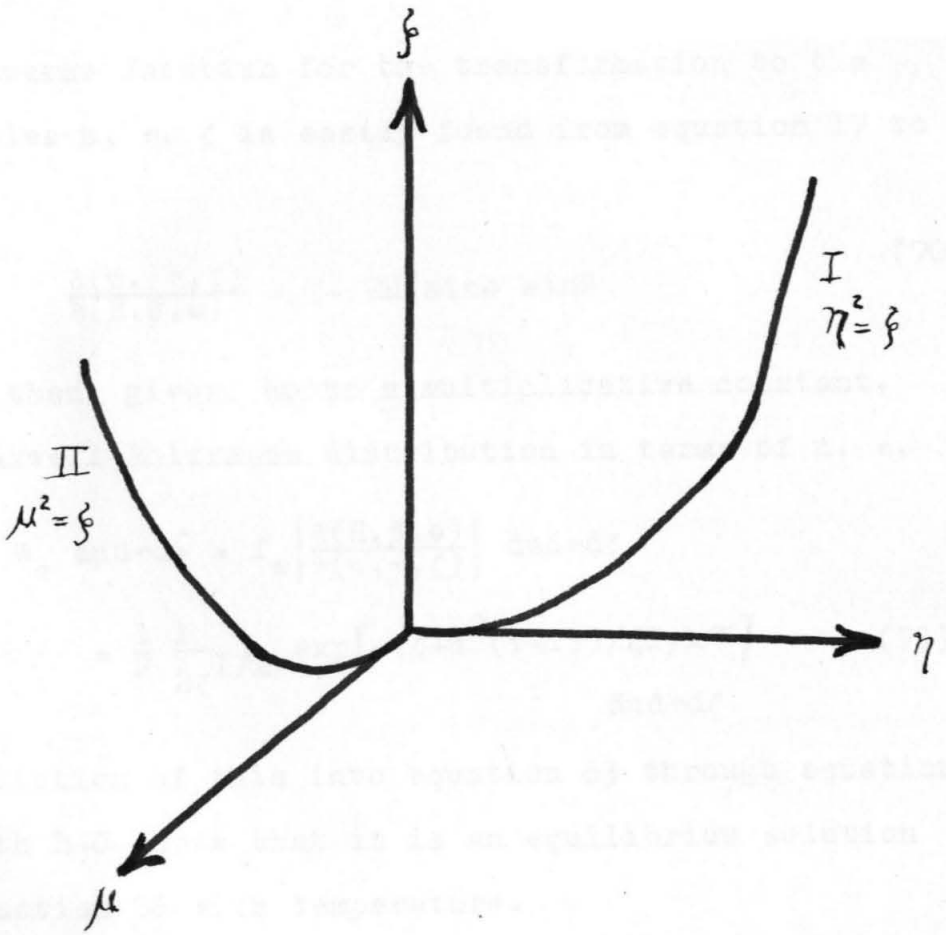

Figure 4. Projections of the bounding surface in μ, η, ζ space.

equilibrium solution for zero magnetic field. Davis and Greenstein found that this distribution for a temperature τ is, up to a multiplicative constant, in terms of β, θ, and H,[16]

$$f_e \, d\beta d\theta dH = \exp\left[-H^2(\sin^2\theta+\gamma\cos^2\theta)/2I\gamma k\tau\right] H^2 \sin\theta \sin\beta \, d\beta d\theta dH$$

The inverse Jacobian for the transformation to the variables μ, η, ζ is easily found from equation 17 to be

$$\frac{\partial(\mu, \eta, \zeta)}{\partial(H,\beta,\theta)} = -2H^3 \sin\theta \sin\beta \tag{70}$$

This, then, gives, up to a multiplicative constant, the Maxwell-Boltzmann distribution in terms of μ, η, ζ as

$$W_e \, d\mu d\eta d\zeta = f_e \left|\frac{\partial(H,\beta,\theta)}{\partial(\mu,\eta,\zeta)}\right| d\mu d\eta d\zeta$$

$$= \frac{1}{2}\frac{1}{2\zeta}^{1/2} \exp\left[-(\zeta+\mu^2(\gamma-1))/2I\gamma k\tau\right] d\mu d\eta d\zeta \tag{71}$$

Substitution of this into equation 63 through equation 66 with B=0 shows that it is an equilibrium solution of equation 66 with temperature.

$$\tau = \left(\frac{m^*}{m}\right)T \tag{72}$$

For the case of elastic collisions, for which $m^*=m$, the distribution is at temperature T, the hydrogen gas temperature. However, for non-elastic collision models, for which $m^*<m$, the distribution, while still Maxwellian,

is characterized by a temperature less than that of the gas cloud. This is to be expected, since in the non-elastic collisions some of the energy is going into internal energy of the grains and is then not available to maintain as high a kinetic energy as otherwise. The hydrogen temperature is not affected, however, since its state is determined overwhelmingly by hydrogen-hydrogen collisions, which are elastic.

7. Final Form for the Equation

The mathematical problem of the orientation process is now that of solving equation 66 with the currents defined by equation 63 through equation 65 for $B \neq 0$ This has the form of a diffusion equation, and we would expect the time dependent solution to be expressible as a sum of terms, each of which decays exponentially with time. Of primary interest for the polarization problem is the term with infinite time constant, that is, the equilibrium solution. The time constant of the next most slowly decaying term is what we will call the relaxation time for the process.

From the algebraic complexity of the current terms, equation 63 through equation 65, we should not expect to be able to write down an exact solution. Certain changes of variable have been found which simplify the equation, or at least put it in a form where

its parts are more familiar, but even after these are carried out, the resulting equation must still be solved by approximate methods. What the simplest form of an equation is, and, therefore, what the best changes of variable are, is, of course, a matter of taste. The author has found a set which he considers most convenient. They were found mostly by a process of trial and error.

The changes of variable are, first, to substitute new independent variables, q,r,s, defined by

$$r = \frac{\eta}{\sqrt{\zeta}} = \cos\beta \qquad -1 \leq r \leq 1$$

$$s = \frac{\mu}{\sqrt{\zeta}} = \cos\theta \qquad -1 \leq s \leq 1 \tag{73}$$

$$q = \frac{\zeta}{m^* c^2 I\gamma} = \frac{H^2}{2\gamma IkT\left(\frac{m^*}{m}\right)} \qquad 0 \leq q < \infty$$

Secondly, substitute the new dependent variable f defined by

$$W = Uf$$

with $$U = \frac{1}{q^{1/2}} e^{-\frac{q}{2}(1+s^2(\gamma-1))} \tag{74}$$

Except for the factor of 1/2 in the exponent, U is the Maxwell-Boltzmann distribution of equation 71. When these changes of variable are carried out, the governing equations, 63 through 66 become

$$\frac{1}{Q}\frac{\partial f}{\partial t} = \frac{1}{q}\left\{\left(\frac{1+\alpha}{2\alpha}\right)(1-s^2)f_{ss} - \left(\frac{1+\alpha}{2\alpha}\right)2sf_s \right. \tag{75}$$

$$+\left(\frac{1+\alpha}{2\alpha}\right)(1-r^2)f_{rr} - \left(\frac{1+\alpha}{2\alpha}\right)2rf_r + 4q^2 f_{qq} + 2\frac{(1+2\alpha)}{\alpha}qf_q$$

$$\left. +\left[\left(\frac{\gamma}{\alpha}+2\right)q - q^2\right]f \right\}$$

$$+ (1-\alpha)\left\{\frac{4}{\alpha}(s-s^3)f_{sq} + \frac{2s^4-3s^2+1}{2\alpha q}f_{ss} + \frac{s^3}{\alpha q}f_s \right.$$

$$\left. - \frac{s^2(1-r^2)}{2\alpha q}f_{rr} + \frac{s^2 r}{\alpha q}f_r + \frac{4s^2 q}{\alpha}f_{qq} \right\}$$

$$+ \left(1-\frac{\gamma^2}{\alpha}\right)qs^2 f$$

$$- \frac{DB^2}{Q}\left\{\frac{1}{2}(\gamma-1)(r^2+1)(s^3-s)f_s - \left[1+s^2(\gamma-1)\right](r^3-r)f_r \right.$$

$$+2\left[1+s^2(\gamma-1)\right](r^2-1)qf_q + \left[\frac{1}{2}(\gamma-1)^2(s^4+s^2)(u^2+1)q \right.$$

$$-(1+2(\gamma-1)s^2)(r^2-1)q$$

$$-\frac{1}{2}(\gamma-1)(s^2+1)(r^2+1) - 2(\gamma-1)^2 s^4 r^2 q$$

$$\left. \left. + 2(\gamma-1)s^2 r^2 - 2\right]f \right\}$$

where $$Q = \frac{\alpha g h}{2\gamma I} = \frac{\alpha h\, m^* n_H}{2\gamma I\, \pi^{1/2}}\left(\frac{2kT}{m}\right)^{1/2} \tag{76}$$

For spherical grains of density 1 gr/cm^3 and radius (10^{-5})cm and hydrogen clouds of $n_H T^{1/2} = 100\ (^\circ K)^{1/2}/cm^3$, which Davis and Greenstein consider typical, this gives $Q = 3x(10^{-13})\ sec^{-1}$.

The fundamental equation is now equation 75. We

will find the approximate solution for the case of small magnetic fields and nearly spherical dust grains.

III. Solution for Small Magnetic Fields and Nearly Spherical Grains

For small magnetic fields and nearly spherical grains we will find a solution in a perturbation series. On the right hand side of equation 75, the terms in the first curly brackets will give the unperturbed equation. This division is used, since, as will be seen, these terms give a self-adjoint operator, thus guaranteeing that its eigenfunctions will form an orthogonal set suitable for expanding the solution of the complete equation. As can be seen, they are also in a form suitable for solution by separation of variables, with the solutions being well known functions. The terms in the second set of curly brackets, multiplied by $(1-\alpha)$, give the perturbation due to the non-sphericity of the grain surface; the next term, containing the factor $(1-\gamma^2/\alpha)$ gives the perturbation due to the non-spherical mechanical properties; the terms in the last curly brackets, multiplied by (DB^2/Q), give the perturbation due to the magnetic field.

Let us denote the unperturbed operator by R_o. Thus

$$R_o[f] = \frac{1}{q}\Big\{\left(\frac{1+\alpha}{2\alpha}\right)(1-s^2)f_{ss} - \left(\frac{1+\alpha}{2\alpha}\right)2sf_s + \left(\frac{1+\alpha}{2\alpha}\right)(1-r^2)f_{rr} - \left(\frac{1+\alpha}{2\alpha}\right)2rf_r + 4q^2 f_{qq} + 2\left(\frac{1+2\alpha}{\alpha}\right)qf_q + \left[\left(\frac{\gamma}{\alpha}+2\right)q - q^2\right]f\Big\} \tag{76}$$

Denote the perturbation by R_1. It will prove convenient to have R_1 broken up into four parts,

$$R_1[f] = R_\gamma[f] + R_B[f] + R_{B\gamma}[f] + R_x[f]$$

with

$$R_\gamma[f] = \frac{(1-\alpha)}{\alpha}\Big\{4(s-s^3)f_{sq} + \frac{2s^4-3s^2+1}{2q}f_{ss} + \frac{s^3}{q}f_s - \frac{s^2(1-r^2)}{2q}f_{rr} + \frac{s^2 r}{q}f_r + 4s^2 q f_{qq}\Big\} + (1-\frac{\gamma^2}{\alpha})qs^2 f$$

$$R_B[f] = \frac{DB^2}{Q}\Big\{(r^3-r)f_r - 2(r^2-1)qf_q + [(r^2-1)q+2]f\Big\} \quad (77)$$

$$R_{B\gamma}[f] = \frac{DB^2}{Q}(\gamma-1)\Big\{-\frac{1}{2}(r^2+1)(s^3-s)f_s + s^2(r^3-r)f_r - 2s^2(r^2-1)qf_q + \Big[2s^2(r^2-1)q + \frac{1}{2}(s^2+1)(r^2+1) - 2r^2s^2\Big]f\Big\}$$

$$R_x[f] = \frac{DB^2}{Q}(\gamma-1)^2\Big[-\frac{1}{2}(s^4+s^2)(r^2+1)q + 2s^4r^2q\Big]f$$

If the usual perturbation theory is to hold, R_o must be self-adjoint. That it is can easily be seen by writing the differential parts as the sum of three operators of the self-adjoint form $w(x)d/dx(p(d/dx)f)$, where $p(x)$ vanishes at the end points of the range of x. This form will be self-adjoint if the scalar product of two functions, say $a(x)$ and $b(x)$ is defined by $(a,b)=\int w^{-1}abdx$.* Expansion will show directly that

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*Call any one of these operators L. Then $(a,Lb)=\int w^{-1}awd/dx(p(d/dx)b)dx=\int p(da/dx)(db/dx)dx$ by integration by parts, since p vanishes at the endpoints of the range of x. This last form is symmetric in a and b, so that $(a,Lb)=(b,La)$, the condition for L being self-adjoint.

$$R_o[f]=\left(\frac{\alpha+1}{2\alpha}\right)\frac{1}{q}\left\{\frac{\partial}{\partial s}\left[(1-s^2)f_s\right]+\frac{\partial}{\partial r}\left[(1-r^2)f_r\right]\right\}$$
$$+q^{-1/2\alpha}\left\{\frac{\partial}{\partial q}(4q^{1+1/2\alpha}f_q)+\left[\left(2+\frac{\gamma}{\alpha}\right)q^{1/2\alpha}-q^{1+1/2\alpha}\right]f\right\}$$

So that $R_o[f]$ is self-adjoint with the scalar product of two arbitrary functions, say f and g, defined with the weight function $q^{1/2\alpha}$ as

$$(f,g) = \int\!\!\int\!\!\int q^{1/2\alpha} fg \, dq\,dr\,ds \tag{78}$$

With our notation of R_o for the unperturbed operator and R_1 for the perturbation, the governing equation for the alignment process, equation 75, is

$$\frac{1}{Q}\frac{\partial g}{\partial t} = R_o[g] + R_1[g]$$

If we look for solutions of the form h(t)f(q,r,s) we get the usual result that

$$h = e^{Q\Lambda t}$$

$$R_o[f] + R_1[f] = \Lambda f$$

These eigenvalues, Λ, will give the spectrum of relaxation times. We will solve these equations by the perturbation series method. In particular, we want to solve the equilibrium equation

$$R_o[f] + R_1[f] = 0 \tag{79}$$

The solution will be given as a series in f_{lmn}, the solutions to the unperturbed equations

$$R_o[f_{lmn}] = \lambda_{lmn} f_{lmn} \tag{80}$$

The solution to equation 79 must be normalized so that the total probability is one. In terms of the original variables this condition is expressed as

$$\iiint W \, d\mu d\eta d\zeta = 1$$

From equation 73, the Jacobian of the transformation to q, r, s is given by

$$\left|\frac{\partial(q,r,s)}{\partial(\mu,\eta,\zeta)}\right|^{-1} = q(m^*c^2 I\gamma)^2 \tag{81}$$

When combined with the substitution of dependent variable, equation 74, this makes the normalization condition

$$\iiint (m^*c^2 I\gamma)^2 \, q^{1/2} \exp\left[-\frac{q}{2}(1+s^2(\gamma-1))\right] f \, dq\,dr\,ds = 1 \tag{82}$$

Most of the effect of non-sphericity of the grains is contained in $R_1[f]$. Not all, however, since the definition of R_o, equation 76 has some coefficients containing α and δ, and the independent variable substitution has a non-spherical factor in the exponent. Because of this, $R_o[f] = 0$ is not generally the equation for spherical grains, and we should not expect the unperturbed eigenvalue corresponding to the equilibrium distribution, λ_{ooo}, to be zero. This turns out to be the case, but upon correction by the perturbation $(f_{ooo}, R_1[f_{ooo}])$ it becomes zero to first order as it

should.

1. Solution of the Unperturbed Equation

We must now solve equation 80. Inspection of the definition of R_o, equation 76, shows that for r and s we have the differential parts of Legendre's equation,

$$(1-s^2)P_l''(s) - 2sP_l'(s) + l(l+1)P_l(s) = 0$$

For the solution to be properly behaved at $s=\pm 1$, l must be an integer, and the solutions are the Legendre polynomials. This suggests that we let

$$f = P_l(s)P_m(r)y(q)$$

where P_l and P_m are the Legendre polynomials. The definition given in the Bateman Manuscript Project[17] will be used. Substituting this in equation 80, gives for y(q) the equation

$$4q^2y'' + 2\left(\frac{1+2\alpha}{\alpha}\right)qy' + \left[\left(\frac{\gamma}{\alpha}+2\right)q-q^2\right]y - \left(\frac{1+\alpha}{2\alpha}\right)by = \lambda qy$$

with $b=l(l+1)+m(m+1)$. If we make the substitution

$$y = q^{\frac{1}{4}\left[\left(\frac{1}{\alpha^2}+4\left(\frac{\alpha+1}{2\alpha}\right)b\right)^{1/2}-\frac{1}{\alpha}\right]}e^{-q/2}z(q)$$

this last equation becomes

$$qz'' + \left[1+\frac{1}{2}\left(\frac{1}{\alpha^2}+4\left(\frac{\alpha+1}{2\alpha}\right)b\right)^{1/2} - q\right]z' + nz = 0$$

where

$$n = \frac{1}{4}\left[\frac{\gamma}{\alpha} - \left(\frac{1}{\alpha^2}+\left(\frac{\alpha+1}{2\alpha}\right)4b\right)^{1/2} - \lambda\right].$$

This is Laguerre's equation, and the properly behaved

solutions are the Laguerre polynomials $L_n^a(q)$ for which $n = 0,1,2,\ldots$

Here $$a = \frac{1}{2}\left[\frac{1}{\alpha^2} + 4\left(\frac{\alpha+1}{2\alpha}\right)b\right]^{1/2}$$

The definition given in the Bateman project will be used.[18] The condition on n gives for the allowed values of λ,

$$\lambda_{lmn} = \frac{\gamma}{\alpha} - \left[\frac{1}{\alpha^2} + \left(\frac{\alpha+1}{2\alpha}\right)4b\right]^{1/2} - 4n \qquad (83)$$

We thus finally arrive at

$$f_{lmn} = N_{lmn}\, q^{\frac{a}{2}-\frac{1}{4\alpha}}\, e^{-q/2}\, P_l(s)P_m(r)L_n^a(q) \qquad (84)$$

with $$a = \frac{1}{2}\left[\frac{1}{\alpha^2} + 4\left(\frac{\alpha+1}{2\alpha}\right)b\right]^{1/2}$$

$$b = l(l+1) + m(m+1)$$

$$l,m,n = 0,1,2,\ldots$$

N_{lmn} is a normalization constant, to be chosen so that

$$(f_{lmn}, f_{lmn}) = 1 \qquad \text{for } l,m,n \neq 0,0,0 \qquad (85)$$

and

$$\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{\infty} (m^*c^2 I\gamma)^2\, q^{1/2}\, e^{-q/2\left[1+s^2(\gamma-1)\right]} f_{ooo}\, dq\,dr\,ds \qquad (86)$$

$$= 1$$

Condition equation 86 makes the f_{lmn} orthonormal, and equation 87 gives agreement to the normalization condition, equation 83, to zero order.

From Bateman[19], we have $\int_{-1}^{1}[P_l(s)]^2\,ds = \frac{2}{2l+1}$ and $\int_0^\infty q^a e^{-q}[L_n^a(q)]^2 dq = \frac{\Gamma(a+n+1)}{n!}$. With the definition of the scalar product, equation 78, these give

$$(N_{lmn})^2 = \left\{\int\int\int_0^\infty q^{1/2\alpha} q^{a-1/2\alpha} e^{-q}[P_l(s)]^2[P_m(r)]^2 [L_n^a(q)]^2\,dq\,dr\,ds\right\}^{-1} = \frac{(2l+1)(2m+1)n!}{4\Gamma(a+n+1)} \quad \text{for } l,m,n \neq 0,0,0 \tag{87}$$

The evaluation of N_{ooo} is given in Appendix II, (II-10).

Equations 83, 84, and 87, together with the definitions in Bateman[17,18], give the solutions of the unperturbed equation, 80. The relaxation times for the process will be determined to zero order by the spectrum, equation 83. The λ next in magnitude to λ_{ooo} will be, for spherical grains for which $\alpha=\gamma=1$,

$$\lambda_{1,0,0} = \lambda_{0,1,0} = -2 \tag{88}$$

From the exponential form for the time dependence, $\exp(Q\lambda t)$, this will give a relaxation time of $1/2Q$. For the typical situation described following equation 76, this will have a value of about $2x(10^{12})$ sec., or about $7x(10^4)$ years.

2. Perturbation Coefficients to be Calculated.

We must now carry out the perturbation solution of equation 79. The lowest order terms of the perturbation series which will be important for the polarization problem will be those of order $(\gamma-1)DB^2/Q$ and $(\alpha-1)DB^2/Q$, that is, those which depend both on the magnetic field and the non-sphericity of the grains. As will be seen, some terms of this order will arise from the second order perturbation terms. Because of the large amount of numerical labor involved, not all of the coefficients of this order will be evaluated, but instead only those will be found which contribute to the polarization parameter, F. The definition of F is given by equation 6,

$$F = -\int\int_{-1}^{1}\left[r^2s^2 + \frac{1}{2}(1-r^2)(1-s^2)\right]p_1(r,s)\,drds \qquad (6)$$

where $p_1(r,s)$ is the deviation from the Maxwell-Boltzmann distribution in the angular variables, r and s, alone.

The equations governing the alignment process have been written in terms of the probability distribution in terms of the variables μ, η, and ζ. In order to see which will be important terms in the perturbation series, let us see how to change this to a probability distribution in the angular variables. First let us change into a distribution over r, s, and q. Using the

transformation equations, 73, the Jacobian of the transformation from μ, η, ζ to r, s, q is easily found to be

$$\frac{\partial(r,s,q)}{\partial(\mu,\eta,\zeta)} = \frac{1}{(m^*c^2 I\gamma)q}$$

Define P(r,s,q) to be the distribution over r, s, and q. Then

$$P(r,s,q)\,drdsdq = W(\mu,\eta,\zeta)\left|\frac{\partial(\mu,\eta,\zeta)}{\partial(r,s,q)}\right| drdsdq$$

$$= (m^*c^2 I\gamma)\, q^4 W\, drdsdq$$

If we also use the transformation from W to f given by equation 74, this becomes

$$P(r,s,q) = (m^*c^2 I\gamma)^2 q^{1/2} \exp\left\{-\frac{q}{2}\left[1+s^2(\gamma-1)\right]\right\} f(r,s,q) \qquad (89)$$

To find the distribution over the angular variables alone, we must integrate P with respect to q. Letting p(r,s) be the distribution over r and s, we have

$$p(r,s) = \int_0^\infty P(r,s,q)\, dq$$

$$= (m^*c^2 I\gamma)^2 \int_0^\infty q^{1/2} \exp\left\{-\frac{q}{2}\left[1+s^2(\gamma-1)\right]\right\} f(r,s,q)\, dq \qquad (90)$$

The perturbation solution will give us $p_1(r,s)$, the deviation of p from the Maxwell-Boltzmann distribution, as a series of Legendre polynomials. From the definition of the Legendre polynomials[17],

$$P_o(r) = 1 \qquad P_2(r) = \tfrac{1}{2}(3r^2 - 1) \tag{91}$$

so that F can be expressed in terms of these as

$$F = -\int_{-1}^{1}\int_{-1}^{1}\left[\tfrac{2}{3}P_2(r)P_2(s)+\tfrac{1}{3}P_o(r)P_o(s)\right] p_1(r,s)\ drds \tag{92}$$

If p_1 expressed as a series,

$$p_1 = \sum_{j,k=0}^{\infty} a_{jk}P_j(r)P_k(s) \tag{93}$$

is substituted in equation 92, the integrals involved will be just integrals of products of Legendre polynomials over their range of orthogonality, so that, because of this orthogonality, only a_{oo} and a_{22} will contribute to F. However, because of the normalization condition on p(r,s) we know immediately that a_{oo} must be zero. The argument runs as follows: The normalization condition is

$$\int_{-1}^{1}\int_{-1}^{1} p(r,s)\ drds = 1$$

From their definitions $p(r,s)=p_e(r,s)+p_1(r,s)$, and $p_e(r,s)$ is the Maxwell-Boltzmann distribution, so that

$$\int_{-1}^{1}\int_{-1}^{1} p_e(r,s,)\ drds = 1$$

Using the expansion, equation 93, we thus have

$$\int_{-1}^{1}\int_{-1}^{1}\left[p_e(r,s) + \sum a_{jk}P_j(r)P_k(s)\right]drds =$$

$$1 + \sum a_{jk}\int_{-1}^{1}\int_{-1}^{1} P_j(r)P_k(s)\,drds = 1$$

Since P_j is orthogonal to $P_o=1$ except if j=o, all the terms in the sum except j=k=o are zero, leaving $a_{oo}=o$. Thus a_{22}, the coefficient of the $P_2(r)P_2(s)$ term in the series expansion of $p_1(r,s)$, is the only coefficient which need be computed.

The calculation of the coefficients of the f_{lmn} in the perturbation solution of equation 79 follows the same method as used in, for example, quantum mechanics. The only difference is in the normalization of the solution. In the usual applications, the solution is normalized to have the integral of its square equal to one, while here we want the total probability to be one, which results in the condition on the integral of the first power of the solution, equation 82. We will give a brief derivation of the necessary formulae and pick out those parts of the solution which contribute to a_{22}.

We wish to solve equation 79, $R_o[f]+R_1[f]=\Lambda f$ with $\Lambda=0$. Suppose that f and Λ are given by the series

$$f = g_o + g_1 + g_2 + \ldots$$

$$\Lambda = \mu_o + \mu_1 + \mu_2 + \ldots$$

with g_o and μ_o independent of, μ_1 and g_1 proportional to the first power of, and g_2 and μ_2 proportional to the second power of the expansion parameter, that is, the strength of R_1. If these are substituted in equation 79 and terms in like powers of the expansion parameter are collected, we get the usual infinite set of equations, the first three of which are

$$R_o[g_o] = \mu_o g_o \tag{94}$$

$$R_1[g_o] + R_o[g_1] = \mu_o g_1 + \mu_1 g_o \tag{95}$$

$$R_1[g_1] + R_o[g_2] = \mu_o g_2 + \mu_1 g_1 + \mu_2 g_2 \tag{96}$$

We are interested in the solution of equation 94 with the smallest value of μ_o. This solution is

$$\mu_o = \lambda_{ooo}$$

$$g_o = f_{ooo}$$

Now expand g_1, in a series of the eigenfunctions of R_o,

$$g_1 = \sum_{lmn} A^{(1)}_{lmn} f_{lmn} \tag{97}$$

If we substitute this expansion in equation 95, and take the scalar product of both sides with f_{lmn} we get the usual results

$$\mu_1 = (f_{ooo}, R_1[f_{ooo}])/(f_{ooo}, f_{ooo}) \quad \text{and} \tag{98}$$

$$A^{(1)}_{lmn} = (f_{lmn}, R_1[f_{ooo}])/(\lambda_{ooo}-\lambda_{lmn}) \quad \text{for } l,m,n \neq 0,0,0$$

Where we have made use of the facts that $R_o[f_{lmn}] = \lambda_{lmn} f_{lmn}$ and, for l, m, n $\neq$ 0,0,0, $(f_{lmn}, f_{lmn})=1$. Also expand g_2 as

$$g_2 = \sum_{lmn} A^{(2)}_{lmn} f_{lmn} \tag{99}$$

Substitute this in equation 96 and take the scalar product of both sides with f_{lmn}. This gives the usual results that, for l,m,n $\neq$ 0,0,0

$$A^{(2)}_{lmn} = \sum_{\substack{i,j,k \\ \neq 0,0,0}} \frac{(f_{lmn}, R_1[f_{ijk}])(f_{ijk}, R_1[f_{ooo}])}{(\lambda_{ooo}-\lambda_{lmn})(\lambda_{ooo}-\lambda_{ijk})}$$

$$+ \frac{(f_{lmn}, R_1[f_{ooo}])}{(\lambda_{ooo}-\lambda_{lmn})}\left[A^{(1)}_{ooo} - \frac{\mu_1}{(\lambda_{ooo}-\lambda_{lmn})}\right] \tag{100}$$

The coefficients $A^{(1)}_{ooo}$ and $A^{(2)}_{ooo}$ would have to be determined from the normalization condition, equation 82.

We have previously, equation 77, divided R_1 up into four pieces, $R_1 = R\gamma + R_{B\gamma} + R_x + R_\beta$. Let us use this division in equation 98 and define

$$b_{lmn} = (f_{lmn}, R_B[f_{ooo}])/(\lambda_{ooo}-\lambda_{lmn})$$

$$c_{lmn} = (f_{lmn}, R_\gamma[f_{ooo}])/(\lambda_{ooo}-\lambda_{lmn})$$

$$d_{lmn} = (f_{lmn}, R_{B\gamma}[f_{ooo}])/(\lambda_{ooo}-\lambda_{lmn}) \qquad (101)$$

$$x_{lmn} = (f_{lmn}, R_x[f_{ooo}])/(\lambda_{ooo}-\lambda_{lmn})$$

Then $A^{(1)}_{lmn} = b_{lmn} + c_{lmn} + d_{lmn} + x_{lmn}$.
From its definition, equation 77, R_x, and therefore x_{lmn}, is proportional to $(\gamma-1)^2DB^2/Q$. We are only going to carry the calculation to lowest order, that is to order $(\gamma-1)DB^2/Q$, and x_{lmn} is therefore of higher order and can be disregarded.

If we similarly use this division of R_1 in equation 100, the expression $(f_{lmn}, R_1[f_{ijk}])\ (f_{ijk}, R_1[f_{ooo}])$ will have four possibilities for each of its factors, thus giving rise to sixteen different kinds of terms. However, using the definitions of equation 77, all but two will be of second or higher power in $(\gamma-1)$, $(\alpha-1)$, or DB^2/Q and can be disregarded. The two remaining terms are $(f_{lmn}, R_\gamma[f_{ijk}])(f_{ijk}, R_B[f_{ooo}])$

$$+ (f_{lmn}, R_B[f_{ijk}])(f_{ijk}, R_\gamma[f_{ooo}]).$$

The factor $[A^{(1)}_{ooo} - \mu_1/(\lambda_{ooo}-\lambda_{lmn})]$ is of first order in the perturbation, so that, similarly, the only contributions

in the final term of equation 100 which cannot be disregarded will be

$$\frac{(f_{lmn},R_\gamma[f_{ooo}])+(f_{lmn},R_B[f_{ooo}])}{(\lambda_{ooo}-\lambda_{lmn})}\left[A^{(1)}_{ooo}-\frac{\mu_1}{\lambda_{ooo}-\lambda_{lmn}}\right].$$

Thus, define

$$e_{lmn}=\sum_{\substack{i,j,k\\ \neq o,o,o}}\frac{(f_{lmn},R_\gamma[f_{ijk}])(f_{ijk},R_B[f_{ooo}])}{(\lambda_{ooo}-\lambda_{lmn})(\lambda_{ooo}-\lambda_{ijk})} \tag{102}$$

$$+\sum_{\substack{i,j,k\\ \neq o,o,o}}\frac{(f_{lmn},R_B[f_{ijk}])(f_{ijk},R_\gamma[f_{ooo}])}{(\lambda_{ooo}-\lambda_{lmn})(\lambda_{ooo}-\lambda_{ijk})}$$

$$+\frac{(f_{lmn},R_\gamma[f_{ooo}])+(f_{lmn},R_B[f_{ooo}])}{(\lambda_{ooo}-\lambda_{lmn})}\left[A^{(1)}_{ooo}-\frac{\mu_1}{\lambda_{ooo}-\lambda_{lmn}}\right]$$

Then finally, to the order to which we are working,

$$A^{(1)}_{lmn}=b_{lmn}+c_{lmn}+d_{lmn}$$

$$A^{(2)}_{lmn}=e_{lmn} \tag{103}$$

Each of these coefficients has an explicit dependence on the perturbation parameters through the definitions of R_γ, R_B, and $R_{B\gamma}$ given by equation 77. These dependences are given by the following table:

Coefficient	Order
b_{lmn}	DB^2/Q
c_{lmn}	$(\gamma-1)$ or $(\alpha-1)$
d_{lmn}	$(\gamma-1)DB^2/Q$
e_{lmn}	$(\gamma-1)DB^2/Q$ or $(\alpha-1)DB^2/Q$

We must now see which of the terms of the pertur-

bation solution will give a contribution to the a_{22} of equation 93. As remarked before, the unperturbed solution, f_{ooo}, is not the Maxwell-Boltzmann solution, since some of the effects of the non-sphericity of the dust grain, even in the absence of a magnetic field, are contained in the perturbation R_γ. The coefficients C_{lmn} represent the first order correction for these effects, and thus contribute to the Maxwell-Boltzmann distribution, $p_e(r,s)$, and therefore not to the $p_1(r,s)$. of equation 93.

The remaining terms of the perturbation solution must now be put into the integral in equation 90 to find their contribution to $p_1(r,s)$. Keeping terms up to first order in $(\gamma-1)$ the integral in equation 90 can be expanded as

$$\int_o^\infty q^{1/2} e^{-q/2} f(r,s,q)dq - \frac{1}{2}(\gamma-1)\int_o^\infty q^{3/2} s^2 e^{-q/2} f(r,s,q)dq \qquad (104)$$

The expansion functions, f_{lmn}, are given by equation 84 as the products of the Legendre polynomials $P_l(s)$ and $P_m(r)$ and a function of q alone. The integration of equation 104 is over q only, so that f_{lmn} contributes $P_l(s)\ P_m(r)$ to $p_1(r,s)$ from the first integral of equation 104 and $(\gamma-1)s^2 P_l(s) P_m(r)$ from the second integral. Since, according to Table 1, d_{lmn} and e_{lmn} already have a factor of $(\gamma-1)$ or $(\alpha-1)$, terms

involving them will not contribute from the second integral to the order to which we are working. It will also turn out that the only non-zero b_{lmn} will be b_{omn} (See the discussion preceeding equation (II-5)), so that the second integral contributes only terms of the form

$$b_{omn}(\gamma-1)s^2 p_m(r) = b_{omn}(\gamma-1)\frac{2P_2(s)+P_o(s)}{3}P_m(r)$$

According to equation 84, we have

$$f_{lmn} = N_{lmn} q^{\frac{a-1}{2}\frac{1}{4\alpha}} e^{-q/2} L_n^a(q)P_l(s)P_m(r) \qquad (84)$$

If this is used in equation 90 as expanded in equation 104, we finally find that the coefficient of the $P_2(s)P_2(r)$ term in $p_1(r,s)$, is given by

$$\frac{a_{22}}{(m^*c^2 I\gamma)^2} = \sum_n (b_{22n}+d_{22n}+e_{22n})N_{22n}\int_o^\infty q^{\frac{a+1}{2}\frac{1}{2}\frac{1}{4\alpha}} e^{-q} L_n^a(q)dq \qquad (105)$$

$$- \frac{2}{3}\frac{(\gamma-1)}{2}\sum_n b_{o2n}N_{o2n}\int_o^\infty q^{\frac{a}{2}+\frac{3}{2}\frac{1}{4\alpha}} e^{-q} L_n^a(q)dq$$

The four sets of coefficients, b_{o2n}, b_{22n}, d_{22n}, and e_{22n}, and then finally a_{22} will be evaluated in Appendix II.

3. Results for Small Magnetic Fields and Nearly Spherical Grains.

The expressions for b_{o2n}, b_{22n}, d_{22n}, and e_{22n}

are given in Appendix II in equations II-13, II-5, II-14, and II-16. The expression for a_{22} is given by equation II-17. It is

$$a_{22} = -(\gamma-1)\frac{DB^2}{Q}\left[\frac{5}{48} + \frac{4}{3\pi^{3/2}}\sum_{n=0}^{\infty}\frac{(n+\frac{1}{2})(n-\frac{9}{2})}{(n+\frac{3}{2})^2(n+\frac{5}{2})(n+\frac{7}{2})}\frac{\Gamma(n-\frac{1}{2})}{n!}\right]$$

$$+(\alpha-1)\frac{DB^2}{Q}\frac{4}{3\pi^{3/2}}\sum_{n=0}^{\infty}\frac{(\frac{3}{2}-n)}{(n+\frac{3}{2})^2(n+\frac{5}{2})(n+\frac{7}{2})}\frac{\Gamma(n+\frac{1}{2})}{n!} \qquad (106)$$

$$= -0.176\,(\gamma-1)\frac{DB^2}{Q} + 0.032\,(\alpha-1)\frac{DB^2}{Q}$$

The physical parameters are contained entirely in $(\gamma-1)$, $(\alpha-1)$, and DB^2/Q, the numerical coefficients being independent constants. Expressions for $(\gamma-1)$ and $(\alpha-1)$ for nearly spherical grains are given by equations 51 through 54.

The effect on the polarization parameter, F, is found from equation 92 and equation 93. They give

$$F = -\int_{-1}^{1}\int_{-1}^{1}\left[\frac{2}{3}P_2(r)P_2(s) + \frac{1}{3}P_0(r)P_0(s)\right]\left\{\sum_{j,k=0}^{\infty}a_{jk}P_j(r)P_k(s)\right\}drds$$

All the integrals except those involving $a_{00}P_0(r)P_0(s)$ and $a_{22}P_2(r)P_2(s)$ will vanish because of the orthogonality of the Legendre polynomials. According to the discussion following equation 93, $a_{00}=0$, so the only term left gives

$$F = -\int_{-1}^{1}\int_{-1}^{1}\frac{2}{3}\left[P_2(r)\right]^2\left[P_2(s)\right]^2 a_{22}drds = -\frac{8}{75}a_{22} \qquad (107)$$

Using the results given by equation 106, we have, to first order in the magnetic field and non-sphericity of the grain,

$$F = +(\gamma-1)\frac{DB^2}{Q}\left[\frac{1}{90} + \frac{32}{225\pi^{3/2}}\sum_{n=0}^{\infty}\frac{(n+\frac{1}{2})(n-\frac{9}{2})}{(n+\frac{3}{2})^2(n+\frac{5}{2})(n+\frac{7}{2})}\frac{\Gamma(n-\frac{1}{2})}{n!}\right] \tag{108}$$

$$-(\alpha-1)\frac{DB^2}{Q}\frac{32}{225\pi^{3/2}}\sum_{n=0}^{\infty}\frac{(\frac{3}{2}-n)}{(n+\frac{3}{2})^2(n+\frac{5}{2})(n+\frac{7}{2})}\frac{\Gamma(n+\frac{1}{2})}{n!}$$

$$= + 0.0188\ (\gamma-1)\frac{DB^2}{Q} - 0.0034\ (\alpha-1)\frac{DB^2}{Q}$$

If we assume that the grain is homogeneous then we may use the relation between $(\gamma-1)$ and $(\alpha-1)$ given by equation 54 to obtain

$$F = + 0.0161\ (\gamma-1)\frac{DB^2}{Q}$$

IV Summary of the Calculation

We will give a brief outline of the structure of the calculation done in the preceding pages.

The equation describing the alignment process is a Fokker-Planck equation.[14] The coefficients in the equation are found from various averages over the assumed probability of dust grain-hydrogen atom collisions. These coefficients are given in terms of the variables $\mu=H\cos\theta$, $\nu=H\cos\beta$, and $\zeta=H^2$ by equations 61 and 62. The governing equation for the probability density $w(\mu,\nu,\zeta)$ in μ, ν, ζ space is then given in equations 63 through 66.

To render the equation more tractable a change of variables is made. The independent variables are changed to

$$r = \cos\beta$$

$$s = \cos\theta$$

$$q = \frac{H^2}{m^* c^2 I\gamma} \tag{73}$$

where c is the characteristic hydrogenatom velocity,

$$c = \left(\frac{2kT}{m}\right)^{1/2} \tag{26}$$

and the dependent variable is changed to f, defined by

$$W = \frac{1}{q^{1/2}} e^{-\frac{q}{2}\left[1+s^2(\gamma-1)\right]} f \tag{74}$$

The result is equation 75.

For the case of small magnetic fields and nearly spherical grains, it is convenient to write the equation as the sum of two parts, $R_o[f] + R_1[f] = \frac{1}{Q}\,\partial f/\partial t$, with R_o, the unperturbed operator defined by equation 76, and $R_1[f]$ the perturbation given by

$$R_1[f] = R_\gamma[f] + R_B[f] + R_{B\gamma}[f] + R_x[f]$$

The perturbations R_γ, R_B, $R_{B\gamma}$, and R_x are defined in equation 77. We are interested in the equilibrium solution,

$$R_o[f] + R_1[f] = 0 \tag{79}$$

This is solved in a perturbation series, the expansion functions being the eigenfunctions of R_o,

$$f_{lmn} = N_{lmn}\, q^{\frac{a}{2}-\frac{1}{4\alpha}}\, e^{-q/2}\, P_l(s) P_m(r) L_n^a(q) \tag{84}$$

The solution up to second order in the perturbation, R_1, is given by equation 97 and 99 as

$$f = f_{ooo} + \sum_{lmn} A^{(1)}_{lmn} + \sum_{lmn} A^{(2)}_{lmn}\, f_{lmn}$$

The formula for the first order coefficients, $A^{(1)}_{lmn}$ is given by equation 98, and for the second order coefficients, $A^{(2)}_{lmn}$, by equation 100.

An analysis is then made of which of the $A^{(1)}_{lmn}$ and $A^{(2)}_{lmn}$ will contribute to the polarization parameter, F, to the order we are working, that is, which will contribute in order $(\gamma-1)DB^2/Q$ or $(\alpha-1)DB^2/Q$. To aid in this, we have the formula equation 90 for changing the solution f(q,r,s) into a probability density, p(r,s), in the space of the angular variables r and s. The results of this analysis are that four sets of coeficients must be found. They are called b_{o2n}, b_{22n}, d_{22n}, and e_{22n}, and are defined by equations 101 and 102.

If we let $p_e(r,s)$ be the distribution of r and s in the absence of a magnetic field, and then expand $p_1(r,s)$ in terms of Legendre polynomials as

$$p(r,s) = p_e(r,s) + \sum_{j,k>0}^{\infty} a_{jk} P_j(r) P_k(s) \tag{5}$$

and (93)

we find that only the a_{oo} term and the a_{22} term will contribute to F, and that further $a_{oo}=0$ so that only a_{22} is needed. (See discussion following equation 93.). In fact the result is that

$$F = -\frac{8}{75} a_{22} \tag{107}$$

This result is true in general. In the perturbation

case that we are working out here, a_{22} is given in terms of b_{o2n}, b_{22n}, d_{22n}, and e_{22n} by equation 105.

The evaluation of b_{o2n}, b_{22n}, d_{22n}, and e_{22n} is done in Appendix II. The results are

$$N_{o2o}b_{o2o} = \frac{DB^2}{Q}\,\frac{1}{2\pi^{1/2}}\,\frac{1}{3} \tag{II-13}$$

$$b_{o2n} = 0 \quad \text{for} \quad n = 0 \tag{II-13}$$

$$b_{22n} = 0 \tag{II-5}$$

$$N_{22n}d_{22n} = (\gamma-1)\frac{DB^2}{Q}\,\frac{1}{2\pi^{1/2}}\,\frac{1}{3}\,\frac{(\frac{5}{2}-n)\,\Gamma(n+\frac{1}{2})}{\Gamma(\frac{3}{2})(n+\frac{3}{2})\,\Gamma(n+\frac{9}{2})} \tag{II-14}$$

$$N_{22n}e_{22n} = (\alpha-1)\frac{DB^2}{Q}\,\frac{1}{2\pi^{1/2}}\,\frac{1}{3}\,\frac{(\frac{3}{2}-n)\,\Gamma(n+\frac{1}{2})}{\Gamma(\frac{3}{2})(n+\frac{3}{2})\,\Gamma(n+\frac{9}{2})} + (\gamma-1)\frac{DB^2}{Q}\,\frac{1}{2\pi^{1/2}}\,\frac{1}{3}\,\frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{3}{2})(n+\frac{3}{2})\,\Gamma(n+\frac{7}{2})} \tag{II-16}$$

The expression for a_{22} is then found to be

$$a_{22} = -(\gamma-1)\frac{DB^2}{Q}\left[\frac{5}{48} + \frac{4}{3\pi^{3/2}}\sum_{n=o}^{\infty}\frac{(n+\frac{1}{2})(n-\frac{9}{2})}{(n+\frac{3}{2})^2(n+\frac{5}{2})(n+\frac{7}{2})}\,\frac{\Gamma(n-\frac{1}{2})}{n!}\right]$$

$$+(\alpha-1)\,\frac{DB^2}{Q}\,\frac{4}{3\pi^{3/2}}\sum_{n=o}^{\infty}\frac{(\frac{3}{2}-n)}{(n+\frac{3}{2})^2(n+\frac{5}{2})(n+\frac{7}{2})}\,\frac{\Gamma(n+\frac{1}{2})}{n!}$$

$$= -0.176\,(\gamma-1)\frac{DB^2}{Q} + 0.032\,(\alpha-1)\frac{DB^2}{Q} \tag{II-17}$$

This results in a final value for F of

$$F = +(\gamma-1)\frac{DB^2}{Q}\left[\frac{1}{90} + \frac{32}{225\pi^{3/2}}\sum_{n=0}^{\infty}\frac{(n+\frac{1}{2})(n-\frac{9}{2})}{(n+\frac{3}{2})^2(n+\frac{5}{2})(n+\frac{7}{2})}\frac{\Gamma(n-\frac{1}{2})}{n!}\right]$$

$$-(\alpha-1)\frac{DB^2}{Q}\frac{32}{225\pi^{3/2}}\sum_{n=0}^{\infty}\frac{(\frac{3}{2}-n)}{(n+\frac{3}{2})^2(n+\frac{5}{2})(n+\frac{7}{2})}\frac{\Gamma(n+\frac{1}{2})}{n!} \qquad (108)$$

$$= +0.0188\ (\gamma-1)\frac{DB^2}{Q} - 0.0034\ (\alpha-1)\frac{DB^2}{Q}$$

V. Conclusion

The alignment of the dust grains, and, thus, their effectiveness as polarizing agents is determined by the equilibrium between two competing processes, the orienting torque due to the paramagnetic relaxation process and the randomizing torques due to collisions with the hydrogen gas. The relative effectiveness of the two processes depends on the ratio of their rate constants.

The rate of the orienting process depends on the internal parameters of the grain and the external magnetic field.

Let V be the volume of the grain,

I be the moment of inertia about the symmetry axis,

γI be the moment of inertia about an axis perpendicular to the symmetry axis,

T_g be the internal temperature of the grain,

and B be the strength of the external magnetic field.

Then the rate constant for alignment of the grains is DB^2 where the constant D is given by equation 9 as

$$DB^2 = \frac{2.5(10^{-12})}{T_g} \frac{V}{I\gamma} \qquad (9)$$

in c.g.s. units and degrees Kelvin. The value of the numerical constant, $2.5(10^{-12})$, depends on the assumed

composition of the grain material. Its value is discussed by Davis and Greenstein.[23] For conditions considered typical, $T_g=10^{o}K$, and spherical grains of radius (10^{-5})cm and density 1 gm/cm^3, $D=6x(10^{-3})$ sec^{-1} gauss^{-2}. A magnetic field of (10^{-5}) gauss would then give $DB^2=6x(10^{-13})$sec.$^{-1}$

The rate of the randomizing process depends on the surface shape and dynamical properties of the grain and the kinetic properties of the hydrogen gas.

Let n_H be the number density of the hydrogen gas,

T be the gas kinetic temperature,

K be Boltzmann's constant,

m be the mass of the gas atoms,

and m^* be the effective mass of the gas atoms (discussed after equation 21).

We also need, in addition, the surface shape parameters, α and h, defined by equation 40 and 42. They are evaluated in equation 48 and 49. For nearly spherical grains they are given in equation 51 as

$$h \doteq \pi a^4\left[\frac{8}{3} + \frac{16}{15}[(b/a)^2 - 1]\right] \qquad (51)$$

$$\alpha h \doteq \pi a^4\left[\frac{8}{3} + \frac{32}{15}[(b/a)^2 - 1]\right]$$

where 2b is the diameter of the shperoid along the rotation axis and 2a the diameter along a perpendicular axis. Then the rate constant for the randomizing pro-

cess is given by equation 76 as

$$Q = \frac{\alpha h}{\gamma I}\left(\frac{m^*}{m}\right) n_H \left(\frac{mkT}{2\pi}\right)^{1/2} \tag{69}$$

For the typical conditions, the same grains as above and $n_H T^{1/2}=100\ (^oK)^{1/2}/cm^3$ and taking $m^*=m$ (corresponding to elastic collisions), $Q=3x(10^{-13})sec.^{-1}$

The type of distribution expected then depends on the ratio

$$\frac{DB^2}{Q} = \frac{2.5(10^{-12})}{T_g}\ \frac{VB^2}{\alpha h n_H}\left(\frac{m}{m^*}\right)\left(\frac{2\pi}{mkT}\right)^{1/2} \tag{109}$$

and the mechanical and surface shape factors, $(\gamma-1)$ and $(\alpha-1)$. For our typical grain and hydrogen conditions $DB^2/Q=2x(10^{10})B^2=2$ for $B=10^{-5}$ gauss. If $(\gamma-1)DB^2/Q \ll 1$, the distribution will be nearly Maxwell-Boltzmann. If $(\gamma-1)DB^2/Q \gg 1$, the distribution will approach complete alignment.

Of primary interest is the polarization parameter, F. For the case carried to completion here, the nearly random situation,

$$F = + 0.0188\ (\gamma-1)\frac{DB^2}{Q} - 0.0034(\alpha-1)\frac{DB^2}{Q} \tag{108}$$

to lowest order in $(\gamma-1)$, $(\alpha-1)$, and DB^2/Q. For uniform spheroids, $(\alpha-1)$ and $(\gamma-1)$ are related by equation 54, and we have

$$F = +0.0161\ (\gamma-1)\frac{DB^2}{Q} \tag{110}$$

We will still maintain the desired order in equation 108 and 110 if we set $\alpha=1$ in the evaulation of DB^2/Q by equation 109. We will then have $\alpha h=\frac{8}{3}\pi a^4$ and $V=\frac{4}{3}\pi a^3$, giving

$$\frac{DB^2}{Q} = \frac{2.5(10^{-12})}{T_g}\left(\frac{m}{m^*}\right)\left(\frac{\pi}{2mk}\right)^{1/2}\frac{B^2}{an_H T^{1/2}} \tag{111}$$

Putting in the numerical value for Boltzmann's constant, K, and the hydrogen atom mass, m, this becomes

$$\frac{DB^2}{Q} = 2.06(10^8)\left(\frac{m}{m^*}\right)\frac{B^2}{aT_g n_H T^{1/2}} \tag{112}$$

and makes equation 110

$$F = +\ 3.32\ (\gamma-1)\left(\frac{m}{m^*}\right)\frac{B^2}{aT_g n_H T^{1/2}}(10^6) \tag{113}$$

We also have obtained a value for the relaxation time. It is given following equation 88 as $1/2Q$. For the typical conditions we have been using as an example, this is about $2x(10^{12})$ sec. If we evaluate Q from equation 69, taking the case of a uniform sphere and putting in the numerical values for m and K, we find the relaxation time is given by

$$1.5(10^{19})\left(\frac{m}{m^*}\right)\frac{a\rho_g}{n_H T^{1/2}} \qquad (114)$$

where ρ_g is the density of a grain. This last result can be compared with the relaxation time obtained by Davis and Greenstein.[2] Their equation 16 is the same as our equation 114 except that they have a numerical coefficient $5(10^{19})$ where we have $1.5(10^{19})(m/m^*)$.

APPENDIX I

We must first justify calculating the moments of the transition probability, E_i and E_{ij}, for a time dt instead of a time Δt. That is, we must justify the use of equations 22 and 23 instead of equations 14 and 15. During a single collision the relative changes of the orientation variables will be much less than one. Thus, we can consider that the orientation variables will be the same for many successive collisions. That is, there will be a time Δt during which many collisions will occur, but also during which the orientation variables of the grain can be considered to remain constant. Therefore, we will have a series of collisions with each having the same transition probability. This is the situation considered by Chandrasekhar in the section of his paper beginning with his equation 94.[21] The result is his equation 103 and the paragraph following it: The first and second moments of the distribution of the total displacement due to a large number, N, of collisions, each having the same transition probability, will be N times the corresponding moments of the distribution for a single collision. This is an example of the usual central limit theorem of statistics. As applied to the situation we are considering this means that (moment for a time Δt) = ($\Delta t/dt$) x (moment for a time dt), so that we may use equations 22 and 23 instead

of equations 14 and 15.

Secondly, we have the additional result from Chandrasekhar's equation 103 that the distribution for a large number of collisions will be Gaussian in form. For this case he finds (his equations 225 and 226[22]) that the third and higher moments of the distribution will be proportional to higher powers of Δt, so that we may use the Fokker-Planck equation, our equation 16.

APPENDIX II

We must evaluate b_{o2n}, b_{22n}, d_{22n}, and e_{22n} according to their definitions, equations 101 and 102. Let us first find $R_B[f_{ooo}]$, $R_\gamma[f_{ooo}]$, and $R_{B\gamma}[f_{ooo}]$.

$$f_{ooo} = N_{ooo} e^{-q/2} \tag{II-1}$$

so that $\frac{\partial}{\partial s} f_{ooo} = \frac{\partial}{\partial r} f_{ooo} = 0$ and $\frac{\partial}{\partial q} f_{ooo} = -\frac{1}{2} f_{ooo}$. If we use this in the definitions of R_B, R_γ, and $R_{B\gamma}$, equation 77, and also use $P_2(s)=\frac{1}{2}(3s^2-1)$, so that $s^2=\frac{1}{3}(2P_2(s)+1)$, we find

$$R_B[f_{ooo}] = N_{ooo} \frac{DB^2}{Q} e^{-q/2}\left\{2 + \frac{4}{3}[P_2(r)-1]\, q\right\} \tag{II-2}$$

$$R_\gamma[f_{ooo}] = N_{ooo}\left(\frac{1-\gamma^2}{3\alpha}\right) e^{-q/2}[1+2P_2(s)]\, q \tag{II-3}$$

$$R_{B\gamma}[f_{ooo}] = N_{ooo} \frac{2}{3}(\gamma-1)\frac{DB^2}{Q}\, e^{-q/2}\left\{1 - P_2(s)P_2(r) + q[2P_2(s)P_2(r)-2P_2(s)+P_2(r)-1]\right\} \tag{II-4}$$

Because of the orthogonality of the Legendre polynomials, we now verify the statement after equation 104 that "the only non-zero b_{lmn} will be b_{omn}." In fact equation II-2 gives us that the only non-zero b_{lmn} are b_{oon} and

b_{o2n}. We thus have

$$b_{22n}=0 \tag{II-5}$$

From equations 101, 102, and 77 we can see that d_{22n} and e_{22n} have factors of either $(\gamma-1)$ or $(\alpha-1)$ explicitly written in their definitions. The contribution of b_{o2n} to a_{22} as given in equation 105 also has an explicit dependence on $(\gamma-1)$. Thus, since we have just found $b_{22n}=0$, all the terms which contribute to a_{22}, according to equation 105, have an explicit dependence on either $(\alpha-1)$ or $(\gamma-1)$. Since we are working to first order in these quantities, we can set $\alpha=\gamma=1$ in all calculations except for these explicit appearances. That is, we get the following simplifications:

Equation 78 becomes

$$(f,g) = \int_{-1}^{1}\int_{-1}^{1}\int_{0}^{\infty} q^{1/2} fg \, dq dr ds \tag{II-6}$$

and equations 83 through 87 become

$$\lambda_{lmn} = 1 - 2a - 4n \tag{II-7}$$

$$f_{lmn} = N_{lmn} q^{\frac{a}{2}-\frac{1}{4}} e^{-q/2} P_l(s) P_m(r) L_n^a(q) \tag{II-8}$$

$$(N_{lmn})^2 = \frac{(2l+1)(2m+1)n!}{4\Gamma(a+n+1)} \quad \text{for } l,m,n \neq 0,0,0 \tag{II-9}$$

with $a = \frac{1}{2}\left[1 + 4l(l+1) + 4m(m+1)\right]^{1/2}$

$$\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{\infty} (m^*c^2 I\gamma)^2 \, q^{1/2} \, e^{-q/2} \, f_{ooo} \, dq\,dr\,ds = 1$$

or $$(m^*c^2 I\gamma)^2 \, N_{ooo} = \left[4\Gamma(\tfrac{3}{2})\right]^{-1} = \frac{1}{2\pi^{1/2}} \qquad \text{(II-10)}$$

The $(m^*c^2 I\gamma)^2$ appearing in equation II-10 will be canceled by its appearance in the left side of equation 105 so that we shall cease using it in either place.

We shall also need the integral given by Bateman[20],

$$\int_{0}^{\infty} q^c e^{-q} L_n^a(q) \, dq = \frac{\Gamma(c+1)\,\Gamma(n+a-c)}{n!\,\Gamma(a-c)} \qquad \text{(II-11)}$$

As a special case of this we have

$$\int_{0}^{\infty} q^c e^{-q} L_n^a(q) \, dq = \begin{cases} \Gamma(a+1) & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases} \qquad \text{(II-12)}$$

which is just the orthogonality relation between L_n^a and $L_o^a = 1$.

If we now combine equations 101, II-2, and II-6 through II-12, we find

$$N_{o2n} b_{o2n} = \begin{cases} \frac{DB^2}{Q} \frac{1}{2\pi^{1/2}} \frac{1}{3} & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases} \qquad \text{(II-13)}$$

Similarly, using equation II-4, we find

$$N_{22n} d_{22n} = (\gamma-1)\frac{DB^2}{Q} \frac{1}{2\pi^{1/2}} \frac{1}{3} \frac{(\frac{5}{2}-n)\Gamma(n+\frac{1}{2})}{\Gamma(\frac{3}{2})(n+\frac{3}{2})\Gamma(n+\frac{9}{2})} \qquad \text{(II-14)}$$

We can also simplify the calculation of e_{22n} from equation 102. Using equations II-2 and II-3, we find, because of the orthogonality of the Legendre polynomials, that

$$(f_{ijk},\ R_B[f_{ooo}])=0 \qquad \text{unless } i,j,k,=o,o,n \text{ or } o,2,n$$

and $(f_{ijk}, R_\gamma[f_{ooo}])=0$ unless $i,j,k = o,o,n$ or $2,o,n$.

From equation 77 and the orthogonality of the Legendre polynomials, we find

$$(f_{22n}, R_\gamma[f_{oon}]) = (f_{22n}, R_B[f_{oon}]) = 0.$$

Because of the application of equation II-12 we also find $(f_{o2k}, R_B[f_{ooo}]) = (f_{2,o,k}, R_\gamma[f_{ooo}]) = 0$ for $k \neq 0$. Thus, equation 102 becomes

$$e_{22n} = \frac{(f_{22n}, R_\gamma[f_{o2o}])(f_{o2o}, R_B[f_{ooo}])}{(\lambda_{ooo}-\lambda_{22n})(\lambda_{ooo}-\lambda_{o2o})} + \frac{(f_{22n}, R_B[f_{2oo}])(f_{2oo}, R_\gamma[f_{ooo}])}{(\lambda_{ooo}-\lambda_{22n})(\lambda_{ooo}-\lambda_{2oo})} \qquad \text{(II-15)}$$

Evaluation of these remaining terms using equations 77 and II-6 through II-12 and expanding the coefficients in equations 77 by

$$\frac{1-\alpha}{\alpha} = -(\alpha-1) + \ldots$$

$$1-\frac{\gamma^2}{\alpha} = (\alpha-1) - 2(\gamma-1) + \ldots$$

finally gives

$$N_{22n}e_{22n} = (\alpha-1)\frac{DB^2}{Q}\,\frac{1}{2\pi^{1/2}}\,\frac{1}{3}\,\frac{(\frac{3}{2}-n)\,\Gamma(n+\frac{1}{2})}{\Gamma(\frac{3}{2})(n+\frac{3}{2})\,\Gamma(n+\frac{7}{2})} \qquad \text{(II-16)}$$

$$+(\gamma-1)\frac{DB^2}{Q}\,\frac{1}{2\pi^{1/2}}\,\frac{1}{3}\,\frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{3}{2})(n+\frac{3}{2})\Gamma(n+\frac{7}{2})}$$

The coefficients $b_{o2n}, b_{22n}, d_{22n}$, and e_{22n} are given by equations II-13, II-5, and II-16. If these are now put into equation 105 and the two integrals appearing there (setting $\alpha=1$) are evaluated by equation II-11), we get

$$a_{22} = -(\gamma-1)\frac{DB^2}{Q}\left[\frac{5}{48} + \frac{4}{3\pi^{3/2}}\sum_{n=o}^{\infty}\frac{(n+\frac{1}{2})(n-\frac{9}{2})}{(n+\frac{3}{2})^2(n+\frac{5}{2})(n+\frac{7}{2})}\,\frac{\Gamma(n-\frac{1}{2})}{n!}\right]$$

$$+(\alpha-1)\frac{DB^2}{Q}\,\frac{4}{3\pi^{3/2}}\sum_{n=o}^{\infty}\frac{(\frac{3}{2}-n)}{(n+\frac{3}{2})^2(n+\frac{5}{2})(n+\frac{7}{2})}\,\frac{\Gamma(n+\frac{1}{2})}{n!} \qquad \text{(II-17)}$$

$$= -0.176\,(\gamma-1)\frac{DB^2}{Q} + 0.032\,(\alpha-1)\frac{DB^2}{Q}$$

REFERENCES

1. Hiltner, Science, 109, 165 (1949)
   Hall, Science, 109, 166 (1949)
2. Davis and Greenstein, Ap. J, 114, 206 (1951)
3. Platt, Ap. J, 123, 486 (1956)
4. Gans, Ann. d. Phys., 37, 881 (1912)
5. See Hiltner, Ap. J. Suppl. No. 24 (1956) for an extensive list of his observations
   and Smith, Ap. J, 124, 43 (1956) for a list of more than 200 stars in the southern Milky Way visible from South Africa.
6. Ref. 2, sec. 4 and 6.
7. Ref. 2, sec. 5.
8. Ref. 2, sec. 6.1, and a more extensive discussion in C.J. Gorter, Paramagnetic Relaxation (New York: Elsevier Pub. Co., 1947)
9. See Davis, Ap. J, 128, 508 (1958) for a rigorous justification for averaging over a nutation in finding the magnetic torque.
10. See Davis, Z. für Astrophysik, 47, 59 (1959) for a qualitative discussion of the wavelength dependence.
11. For instance, Lowell Observatory Bull., No. 105, 4, 264 (1960)
12. Ref. 2, sec. 6 and 7.

13. Eq. (10), (11), and (12) are ref. 2, eq. (82), (85) and a combination of (74) and (84).

14. Chandrasekhar, Rev. Mod. Phys., 15, 1 (1943). For the Fokker-Planck equation see especially pg. 31 through 44.

15. Ref. 2, pg. 212 or pg. 231, eq. (81)

16. Ref. 2 pg. 237 through 239, but note missing exponent in eq. (121).

17. Bateman Manuscript Project, Higher Transcendental Functions, Vol. II (New York: McGraw Hill Book Co., 1953) pg. 179.

18. Ref. 17, pg. 188.

19. Ref. 17, pg. 179 and 188.

20. Bateman Manuscript Project, Tables of Integral Transforms, Vol. II(New York: McGraw Hill Book Co., 1954) pg. 292.

21. Ref. 14, pg. 15.

22. Ref. 14, pg. 33.